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ZADATAK 1 Razmotrite sustav čija je valna funkcija jednaka

$$\psi(\theta,\phi)=\frac{1}{2}Y_0^0(\theta,\phi)+\frac{1}{\sqrt{3}}Y_1^1(\theta,\phi)+\frac{1}{2}Y_1^{-1}(\theta,\phi)+\frac{1}{\sqrt{6}}Y_2^2(\theta,\phi)$$

(a) Je li ψ normalizirana?
(b) Je li ψ svojstvena funkcija za $\mathbf{L}^2$ i L_z?
(c) Izračunajte $L_\pm\psi$ i $\langle\psi| L_\pm |\psi\rangle$.
(d) Ako mjerimo z-komponentu orbitalnog angularnog momenta, nađite vjerojatnost da izmjerimo 0, $\hbar$, $-\hbar$ i $2\hbar$.
***Uputa*:** za operatore podizanja i spuštanja L_+ i L_- vrijede uobičajene relacije

$$L_\pm = L_x \pm iL_y$$

$$L_\pm Y_l^m(\theta,\phi)=\hbar\sqrt{l(l+1)-m(m\pm1)}Y_l^{m\pm1}(\theta,\phi)$$

ZADATAK 2 (a) Čestica spina 1 i čestica spina 2 miruju u konfiguraciji u kojoj je ukupni spin 3 i z-komponenta ukupnog spina 1, u jedinicama $\hbar$. Ako mjerimo z-komponentu spina čestice 2, koje vrijednosti ćete dobiti i s kojim vjerojatnostima?
(b) Elektron sa spinom $m_s = -1/2$ nalazi se u stanju ψ_{510} u vodikovom atomu. Ako biste mogli mjeriti ukupni angularni moment elektrona (bez spina protona), koje vrijednosti biste dobili i s kojim vjerojatnostima?

ZADATAK 3 Nađite energiju osnovnog stanja ($l = 0$) i valnu funkciju za Hulthénov potencijal

$$V(r)=-\frac{V_0e^{-\alpha r}}{1-e^{-\alpha r}}$$

gdje je $V_0 > 0$.
***Uputa*:** potražite rješenje radijalne Schrödingerove jednadžbe u obliku

$$\psi(r)=\frac{A}{r}e^{-\gamma r}\left(1-e^{-\alpha r}\right)$$

gdje je A konstanta. Koji uvjet mora zadovoljavati konstanta γ?

1.

(a) Treba provjeriti je li

$$\int |\psi|^2 d\Omega = 1$$

Sferni harmonici su ortonormirane funkcije

$$\int d\Omega \, Y_l^{m*} Y_{l'}^{m'} = \delta_{ll'} \delta_{mm'}$$

pa je, na primjer

$$\int Y_1^{1*} Y_2^{2} d\Omega = \delta_{21} \delta_{21} = 0$$

$$\int Y_1^{1*} Y_1^{1} d\Omega = \delta_{11} \delta_{11} = 1$$

Imamo,

$$\int |\psi|^2 d\Omega = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{6}}\right)^2$$

$$= \frac{1}{4} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} = \frac{3+4+3+2}{12} = \underline{\underline{1}}$$

ψ je normalizirana.

(b)

$$\vec{L}^2 \psi = \frac{1}{2} \vec{L}^2 Y_0^0 + \frac{1}{\sqrt{3}} \vec{L}^2 Y_1^1 + \frac{1}{2} \vec{L}^2 Y_1^{-1} + \frac{1}{\sqrt{6}} \vec{L}^2 Y_2^2$$

$$= \frac{1}{2} \cdot 0 + \frac{1}{\sqrt{3}} \cdot 2\hbar^2 Y_1^1 + \frac{1}{2} 2\hbar^2 Y_1^{-1} + \frac{1}{\sqrt{6}} 6\hbar^2 Y_2^2$$

$$\neq \lambda_l \psi$$

ψ nije svojstvena za $\vec{L}^2$.

$$L_z \psi = \frac{1}{2}\cdot 0 + \frac{1}{\sqrt{3}}\cdot \hbar Y_1^1 + \frac{1}{2}(-1)\hbar Y_1^{-1} + \frac{1}{\sqrt{6}} 2\hbar Y_2^2$$

$$\neq \lambda_m \psi$$

ψ nije svojstvena za L_z.

(c)

$$L_+ \psi = \frac{1}{2} L_+ Y_0^0 + \frac{1}{\sqrt{3}} L_+ Y_1^1 + \frac{1}{2} L_+ Y_1^{-1} + \frac{1}{\sqrt{6}} L_+ Y_2^2$$

$$L_+ Y_0^0 = 0$$

NE POSTOJI

$$L_+ Y_1^1 = \hbar \sqrt{\underbrace{2-2}_{=0}}\, Y_1^2 = 0$$

$$L_+ Y_1^{-1} = \hbar \sqrt{2 - (-1)\underbrace{(-1+1)}_{=0}}\, Y_1^0 = \sqrt{2}\hbar Y_1^0$$

NE POSTOJI

$$L_+ Y_2^2 = \hbar \sqrt{2\cdot 3 - 2\cdot 3}\, Y_2^3 = 0$$

Prema tome,

$$L_+ \psi = \frac{1}{2}\cdot \sqrt{2}\hbar Y_1^0$$

$$L_- \psi = \frac{1}{2} L_- Y_0^0 + \frac{1}{\sqrt{3}} L_- Y_1^1 + \frac{1}{2} L_- Y_1^{-1} + \frac{1}{\sqrt{6}} L_- Y_2^2$$

$$L_- Y_0^0 = 0$$

$$L_- Y_1^1 = \hbar \sqrt{2-0}\, Y_1^0 = \sqrt{2}\, Y_1^0 \hbar$$

NE POSTOJI

$$L_- Y_1^{-1} = \hbar \sqrt{2-(-1)(-2)}\, Y_1^{-2} = 0$$

$$L_- Y_2^2 = \hbar \sqrt{2\cdot 3 - 2\cdot(1)}\, Y_2^1 = 2\hbar Y_2^1$$

Dobyemo

$$L_-\psi = \frac{\hbar}{\sqrt{3}}\sqrt{2}\,Y_1^0 + \frac{1}{\sqrt{6}}\cdot 2\hbar\, Y_2^1$$

Prosječne vrijednosti

$$\langle\psi|L_+|\psi\rangle = \langle\psi|\cdot(L_+|\psi\rangle) = \int d\Omega\,\psi^*\cdot\frac{\sqrt{2}}{2}\hbar\,Y_1^0$$
$$= 0$$

zbog relacija ortonormiranosti.

$$\langle\psi|L_-|\psi\rangle = \int d\Omega\,\psi^*\cdot\left(\hbar\sqrt{\frac{2}{3}}\,Y_1^0 + \frac{2}{\sqrt{6}}\hbar\,Y_2^1\right)$$
$$= 0$$

(d) Vjerojatnost za mjerenje L_z u jedinicama od $\hbar$

$m=0$; $\left|\frac{1}{2}\right|^2 = \frac{1}{4}$

$m=1$; $\left|\frac{1}{\sqrt{3}}\right|^2 = \frac{1}{3}$

$m=-1$; $\left|\frac{1}{2}\right|^2 = \frac{1}{4}$

$m=2$; $\left|\frac{1}{\sqrt{6}}\right|^2 = \frac{1}{6}$

$\boxed{2.}$

(a) Tablica 2×1

Početno stanje:

$$|j=3, m=1\rangle = \frac{1}{\sqrt{15}}|m_1=2, m_2=-1\rangle + \sqrt{\frac{8}{15}}|m_1=1, m_2=0\rangle + \sqrt{\frac{2}{5}}|m_1=0, m_2=1\rangle$$

Kako sam zapisao, čestica 1 ima spin 2 pa je vjerojatnost

$$P = \left|\frac{1}{\sqrt{15}}\right|^2 = \frac{1}{15}$$

(b) Elektron: $m_s = -\frac{1}{2}$; $s = \frac{1}{2}$

$n = 5$

$l = 1$

$m_l = 0$

Prema tome, elektron je u stanju

$$|m_l = 0, m_s = -\tfrac{1}{2}\rangle$$

kojeg tražimo u tablici $1 \times 1/2$ zbog $l=1$, $s=\frac{1}{2}$.

$$|m_l=0, m_s=-\tfrac{1}{2}\rangle = \sqrt{\frac{2}{3}}\,|j=\tfrac{3}{2}, m=-\tfrac{1}{2}\rangle + \sqrt{\frac{1}{3}}\,|j=\tfrac{1}{2}, m=-\tfrac{1}{2}\rangle$$

$j = \frac{\hbar}{2}$; $P_{1/2} = \left|\frac{1}{\sqrt{3}}\right|^2 = \frac{1}{3}$

$j = \frac{3}{2}\hbar$; $P_{3/2} = \left|\sqrt{\frac{2}{3}}\right|^2 = \frac{2}{3}$

$\boxed{3.}$

Radijalna Schrödingerova jednadžba za $l=0$ glasi

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} - V_0\frac{e^{-\alpha r}u}{1-e^{-\alpha r}} = Eu \qquad (*)$$

gdje je $E<0$, $E=-|E|$ za vezano stanje. Valna funkcija zadana u zadatku je ustvari

$$\psi(r) \propto R(r)$$

pa je zbog

$$R(r)=\frac{u(r)}{r} \propto \frac{A}{r}e^{-\gamma r}\left(1-e^{-\alpha r}\right)$$

funkcija $u(r)$ jednaka

$$u(r) = C\cdot e^{-\gamma r}\left(1-e^{-\alpha r}\right)$$

gdje je C normalizacijska konstanta. Uvrstimo u $(*)$.

$$\begin{aligned}\frac{du}{dr} &= C(-\gamma)e^{-\gamma r}\left(1-e^{-\alpha r}\right) + Ce^{-\gamma r}\cdot(\alpha)e^{-\alpha r}\\ &= -C\gamma e^{-\gamma r} + C\gamma e^{-(\alpha+\gamma)r} + C\alpha e^{-(\alpha+\gamma)r}\\ &= -C\gamma e^{-\gamma r} + C(\gamma+\alpha)e^{-(\gamma+\alpha)r}\end{aligned}$$

$$\frac{d^2u}{dr^2} = C\gamma^2 e^{-\gamma r} - C(\gamma+\alpha)^2 e^{-(\gamma+\alpha)r}$$

Imamo

$$\left[C\gamma^2 e^{-\gamma r} - C(\gamma+\alpha)^2 e^{-(\gamma+\alpha)r}\right]\left(1-e^{-\alpha r}\right) + \beta^2\cdot e^{-\alpha r}\cdot Ce^{-\gamma r}\left(1-e^{-\alpha r}\right) - \varkappa^2\cdot Ce^{-\gamma r}\left(1-e^{-\alpha r}\right)\left(1-e^{-\alpha r}\right) = 0$$

gdje su

$$\zeta^2 = \frac{2mV_0}{\hbar^2} \quad ; \quad \varkappa^2 = \frac{2m|E|}{\hbar^2}$$

Konstantu C možemo skratiti, kao i $(1-e^{-\alpha r})$. Ostaje

$$\gamma^2 e^{-\gamma r} - (\gamma+\alpha)^2 e^{-(\gamma+\alpha)r} + \zeta^2 e^{-(\gamma+\alpha)r} - \varkappa^2 e^{-\gamma r}(1-e^{-\alpha r}) = 0$$

Možemo skratiti $e^{-\gamma r}$:

$$(\gamma^2 - \varkappa^2) + e^{-\alpha r}\left[\varkappa^2 + \zeta^2 - (\gamma+\alpha)^2\right] = 0$$

Budući su 1 i $e^{-\alpha r}$ linearno nezavisne funkcije, mora biti

$$\gamma^2 - \varkappa^2 = 0 \quad \Rightarrow \quad \gamma = \varkappa$$

$$\varkappa^2 + \zeta^2 - (\gamma+\alpha)^2 = 0$$

Ovo su dvije jednadžbe za γ i za E, odnosno, $\varkappa$.

Dalje je

$$\varkappa^2 + \zeta^2 - (\varkappa+\alpha)^2 = \underbrace{\cancel{\varkappa^2} + \zeta^2 - \cancel{\varkappa^2} - 2\varkappa\alpha + \alpha^2}_{=0}$$

$$\boxed{\varkappa = \frac{\zeta^2}{2\alpha} - \frac{\alpha}{2}}$$

Prema tome, energija osnovnog stanja ($l=0$)

$$E = -|E| = -\frac{\hbar^2\varkappa^2}{2m} = -\frac{\hbar^2}{2m}\left(\frac{mV_0}{\hbar^2\alpha} - \frac{\alpha}{2}\right)^2$$

$$\gamma = \frac{\zeta^2}{2\alpha} - \frac{\alpha}{2} = \frac{mV_0}{\hbar^2\alpha} - \frac{\alpha}{2}$$