MATEMATIČKE METODE FIZIKE I

Drugi kolokvij 2. 2. 2024.

ZADATAK 1 Nađite divergenciju i rotaciju vektorskog polja

$$\mathbf{v} = \rho(2+\sin^2\phi)\mathbf{e}_\rho + \rho\sin\phi\cos\phi\,\mathbf{e}_\phi + 3z\,\mathbf{e}_z$$

ZADATAK 2 Pokažite da vektori Frenetovog trobrida krivulje

$$\mathbf{r}(t) = e^t\cos t\,\mathbf{e}_x + e^t\sin t\,\mathbf{e}_y + e^t\mathbf{e}_z$$

zatvaraju konstantne kutove sa z-osi.

ZADATAK 3 Izračunajte krivuljni integral

$$\oint_C \left[(\mathrm{e}^x\,y + \cos x\sin y)\,\mathrm{d}x + (\mathrm{e}^x + \sin x\cos y)\,\mathrm{d}y\right]$$

po elipsi $x^2/a^2 + y^2/b^2 = 1$.

ZADATAK 4 (a) Pokažite da plošni integral

$$\int_\sigma \left(4xyz\,dxdy - 2x^2y\,dydz - 3xz^2\,dxdz\right)$$

ne ovisi o površini $S(\sigma)$ nego samo o njenoj rubnoj krivulji C.
(b) Transformirajte plošni integral pod (a) u krivuljni integral po rubnoj krivulji C te ga izračunajte ako je rubna krivulja zadana jednadžbama:

$$x^2 + y^2 = R^2, \quad x + z = 0$$

***Uputa*:** napišite zadani integral u obliku

$$\int_\sigma \mathbf{A}\cdot\mathbf{n}dS$$

te provjerite je li $\mathbf{A}$ solenoidalno polje.

ZADATAK 5 Izračunajte plošni integral vektorskog polja $\mathbf{A}$

$$\mathbf{A}(\mathbf{r}) = (x^2+y^2+z^2)(x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z)$$

po sferi $R^2 = x^2 + y^2 + z^2$. Računajte direktno i nakon toga računajte pomoću teorema o divergenciji.

8.3

Divergencija u cilindričkim koordinatama:

$$\vec{v} = v_\rho \vec{e}_\rho + v_\varphi \vec{e}_\varphi + v_z \vec{e}_z$$

$$\operatorname{div} \vec{v} = \vec{\nabla}\cdot\vec{v} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho v_\rho) + \frac{1}{\rho}\frac{\partial v_\varphi}{\partial\varphi} + \frac{\partial v_z}{\partial z}$$

$$\vec{v} = \underbrace{\rho(2+\sin^2\varphi)}_{v_\rho}\vec{e}_\rho + \underbrace{\rho\sin\varphi\cos\varphi}_{v_\varphi}\,\vec{e}_\varphi + \underbrace{3z}_{v_z}\vec{e}_z$$

$$\vec{\nabla}\cdot\vec{v} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho^2(2+\sin^2\varphi)\right) + \frac{1}{\rho}\frac{\partial}{\partial\varphi}\Big(\rho\underbrace{\sin\varphi\cos\varphi}_{\frac{1}{2}\sin 2\varphi}\Big) + \frac{\partial}{\partial z}(3z)$$

$$= \frac{1}{\rho}\left(2\rho(2+\sin^2\varphi)\right) + \frac{1}{\rho}\cdot\rho\cdot\frac{1}{2}\cos 2\varphi\cdot 2 + 3$$

$$= 2(2+\sin^2\varphi) + \cos 2\varphi + 3$$

$$= 4 + 2\sin^2\varphi + \cos^2\varphi - \sin^2\varphi + 3$$

$$= 8$$

Rotacija u cilindričkim koordinatama:

$$\operatorname{rot}\vec{v} = \vec{\nabla}\times\vec{v} = \left(\frac{1}{\rho}\frac{\partial v_z}{\partial\varphi} - \frac{\partial v_\varphi}{\partial z}\right)\vec{e}_\rho + \left(\frac{\partial v_\rho}{\partial z} - \frac{\partial v_z}{\partial\rho}\right)\vec{e}_\varphi$$

$$+ \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho V_\varphi) - \frac{1}{\rho}\frac{\partial v_\rho}{\partial\varphi}\right)\vec{e}_z$$

$$\operatorname{rot}\vec{v} = \left[\frac{1}{\rho}\cdot\frac{\partial}{\partial\varphi}(3z) - \frac{\partial}{\partial z}(\rho\sin\varphi\cos\varphi)\right]\vec{e}_\rho + \left[\frac{\partial}{\partial z}\left(\rho(2+\sin^2\varphi)\right) - \frac{\partial}{\partial\rho}(3z)\right]\vec{e}_\varphi$$

$$+ \left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho^2\sin\varphi\cos\varphi\right) - \frac{1}{\rho}\frac{\partial}{\partial\varphi}\left(\rho(2+\sin^2\varphi)\right)\right]\vec{e}_z$$

$$= 0\,\vec{e}_\rho + 0\,\vec{e}_\varphi + \Big[2\sin\varphi\underbrace{\cos\varphi - 2\sin\varphi\cos\varphi}_{=0}\Big]\vec{e}_z$$

$$= 0$$

Polje $\vec{v}$ je konzervativno (bezvrtložno).

$\boxed{2.}$ Nađimo $\vec{t}$, $\vec{b}$ i $\vec{n}$.

$$\vec{t} = \frac{\dot{\vec{r}}}{\|\dot{\vec{r}}\|}$$

$$\dot{\vec{r}} = e^t(\cos t\, \vec{e}_x + \sin t\, \vec{e}_y + \vec{e}_z) + e^t(-\sin t\, \vec{e}_x + \cos t\, \vec{e}_y)$$

$$= e^t(\cos t - \sin t)\vec{e}_x + e^t(\sin t + \cos t) + e^t\vec{e}_z$$

$$\|\dot{\vec{r}}\|^2 = (e^t)^2\Big[\underbrace{(\cos t - \sin t)^2}_{\cos^2 + \sin^2 t - 2\sin t \cos t} + \underbrace{(\sin + \cos t)^2}_{\sin^2 + \cos^2 t + 2 \sin t \cos t} + 1\Big]$$

$$= (e^t)^2 \cdot 3 \Rightarrow \|\dot{\vec{r}}\| = 3e^t$$

Imamo,

$$\vec{t} = \frac{1}{\sqrt{3}}\left[(\cos t - \sin t)\vec{e}_x + (\sin t + \cos t)\vec{e}_y + \vec{e}_z\right]$$

No,

$$\vec{t}\cdot\vec{e}_z = \frac{1}{\sqrt{3}}$$

pa smo za tangentu dokazali! Računamo binormalu

$$\vec{b} = \pm\frac{\dot{\vec{r}} \times \ddot{\vec{r}}}{\|\dot{\vec{r}} \times \ddot{\vec{r}}\|}$$

$$\ddot{\vec{r}} = \frac{d}{dt}\dot{\vec{r}} = \frac{d}{dt}\left\{e^t\left[(\cos t - \sin t)\vec{e}_x + (\sin t + \cos t)\vec{e}_y + \vec{e}_z\right]\right\}$$

$$= e^t\left[(\cos t - \sin t)\vec{e}_x + (\sin t + \cos t)\vec{e}_y + \vec{e}_z\right]$$

$$+ e^t\cdot\left[(-\sin t - \cos t)\vec{e}_x + (\cos t - \sin t)\vec{e}_y\right]$$

$$\ddot{\vec{r}} = e^{t} \cdot (-2\sin t)\,\vec{e}_x + 2\cos t\,\vec{e}_y + \vec{e}_z$$

$$\dot{\vec{r}} \times \ddot{\vec{r}} = \det\begin{pmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \cos t - \sin t & \sin t + \cos t & 1 \\ -2\sin t & 2\cos t & 1 \end{pmatrix} e^{2t}$$

$$= \vec{e}_x \cdot \Big[(\sin t + \cos t) - 2\cos t\Big] - \vec{e}_y\Big[(\cos t - \sin t) + 2\sin t\Big]$$

$$+ \vec{e}_z\Big[2\cos t(\cos t - \sin t) + 2\sin t(\sin t + \cos t)\Big]$$

$$= \vec{e}_x \cdot (\sin t - \cos t) - \vec{e}_y(\cos t + \sin t)$$

$$+ \vec{e}_z \cdot (\underbrace{2\cos^2 t + 2\sin^2 t}_{2})$$

Iznos ovog vektora je

$$e^{2t}\left[(\sin t - \cos t)^2 + (\cos t + \sin t)^2 + 4\right]^{1/2} = \sqrt{6}\cdot e^{2t}$$

Prema tome,

$$\vec{b} = + \frac{\dot{\vec{r}} \times \ddot{\vec{r}}}{\|\dot{\vec{r}} \times \ddot{\vec{r}}\|} = + \frac{1}{\sqrt{6}}\Big[\vec{e}_x(\sin t - \cos t)$$

$$- \vec{e}_y(\cos t + \sin t) + 2\vec{e}_z\Big]$$

$$\vec{b}\cdot\vec{e}_z = \frac{2}{\sqrt{6}}$$

Na kraju, računamo $\vec{n} = \vec{b} \times \vec{t}$

$$\vec{n} = \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{6}} \det\begin{pmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \sin t - \cos t & -\cos t - \sin t & 2 \\ \cos t - \sin t & \sin t + \cos t & 1 \end{pmatrix}$$

$$\vec{n} = \frac{1}{3\sqrt{2}} \Big[\vec{e}_x \cdot (-\cos t - \sin t - 2\sin t - 2\cos t)$$

$$- \vec{e}_y (\sin t - \cos t - 2\cos t + 2\sin t)$$

$$+ \vec{e}_z \Big[\underbrace{(\sin t - \cos t)(\sin t + \cos t)}_{\sin^2 t - \cos^2 t} + \underbrace{(\sin t + \cos t)(\cos t - \sin t)}_{\cos^2 t - \sin^2 t} \Big] \Big]$$

$$\vec{n} = \frac{1}{3\sqrt{2}} \left[-3\vec{e}_x \cdot (\sin t + \cos t) - 3\vec{e}_y (\sin t - \cos t) \right]$$

$$= -\frac{1}{\sqrt{2}} \left[(\sin t + \cos t)\vec{e}_x + (\sin t - \cos t)\vec{e}_y \right]$$

pa je $\quad \vec{n} \cdot \vec{e}_z = 0$

$\boxed{10.75}$

Upotrijebit ćemo Greenov teorem

$$\oint_C P\,dx + Q\,dy = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

$$P = e^x y + \cos x \sin y$$

$$Q = e^x + \sin x \cos y$$

$$\frac{\partial Q}{\partial x} = e^x + \cos x \cos y$$

$$\frac{\partial P}{\partial y} = e^x + \cos x \cos y$$

Imamo,

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$$

pa je

$$\oint_C P\,dx + Q\,dy$$

po bilo kojoj krivulji C jednak nuli. Znači, radi se o konzervativnom polju.

$\boxed{4.}$

(a) Navedeni integral moramo napisati u obliku

$$\int \vec{A}\cdot\vec{n}\,dS$$

$$\vec{n} = n_x\vec{e}_x + n_y\vec{e}_y + n_z\vec{e}_z \quad ; \quad n_x^2+n_y^2+n_z^2 = 1$$

Imamo

$$\int (-2x^2y\,dydz - 3xz^2dxdz + 4xyz\,dxdy)$$

$$= \int (-2x^2y\,n_x - 3xz^2n_y + 4xyz\,n_z)\,dS$$

Na primjer, $dxdy = n_z dS = \cos\gamma\, dS$

gdje γ kut između normale i $\vec{e}_z$ vektora

Vidimo da je

$$\vec{A} = -2x^2y\,\vec{e}_x - 3xz^2\vec{e}_y + 4xyz\,\vec{e}_z$$

Je li $\vec{A}$ solenoidalno?

$$\vec{\nabla}\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$= -4xy + 0 + 4xy = 0$$

Jest! Prema tome, postoji vektorsko polje $\vec{b}$ za kojeg vrijedi

$$\vec{A} = \vec{\nabla}\times\vec{b}$$

Sada treba pronaći to polje:

$$\vec{\nabla}\times\vec{b} = \det\begin{pmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \partial_x & \partial_y & \partial_z \\ b_x & b_y & b_z \end{pmatrix} = \vec{e}_x\cdot\left(\frac{\partial b_z}{\partial y} - \frac{\partial b_y}{\partial z}\right)$$

$$- \vec{e}_y\left(\frac{\partial b_z}{\partial x} - \frac{\partial b_x}{\partial z}\right) + \vec{e}_z\left(\frac{\partial b_y}{\partial x} - \frac{\partial b_x}{\partial y}\right)$$

Prema tome,

1. $$-2x^2y = \frac{\partial b_z}{\partial y} - \frac{\partial b_y}{\partial z}$$

2. $$-3xz^2 = -\left(\frac{\partial b_z}{\partial x} - \frac{\partial b_x}{\partial z}\right)$$

3. $$4xyz = \frac{\partial b_y}{\partial x} - \frac{\partial b_x}{\partial y}$$

Iz jednadžbe 1.

$$\frac{\partial b_z}{\partial y} = \frac{\partial b_y}{\partial z} - 2x^2y \quad / \int dy$$

$$b_z = -x^2y^2 + \int \frac{\partial b_y}{\partial z}\,dy$$

Uvrstimo u 2.

$$+3xz^2 = -2xy^2 + \int \frac{\partial^2 b_y}{\partial x\partial z}\,dy - \frac{\partial b_x}{\partial z}.$$

Odavde je

$$\frac{\partial b_x}{\partial z} = -2xy^2 - 3xz^2 + \int \frac{\partial^2 b_y}{\partial x\partial z}\,dy \quad / \int dz$$

$$b_x = -2xy^2z - xz^3 + \int \frac{\partial b_y}{\partial x}\,dy$$

Uvrstimo u jednadžbu 3.

$$4xyz = \frac{\partial b_y}{\partial x} - \left(-4xyz - 0 + \frac{\partial b_y}{\partial x}\right)$$

$$0 = 0$$

Vidimo da je b_y proizvoljna : odaberimo

$$b_y = 0$$

Imamo,

$$b_x = -2xy^2z - xz^3$$

$$b_y = 0$$

$$b_z = -x^2y^2$$

Sada integral po plohi možemo transformirati na integral po krivulji po Stokesovom teoremu

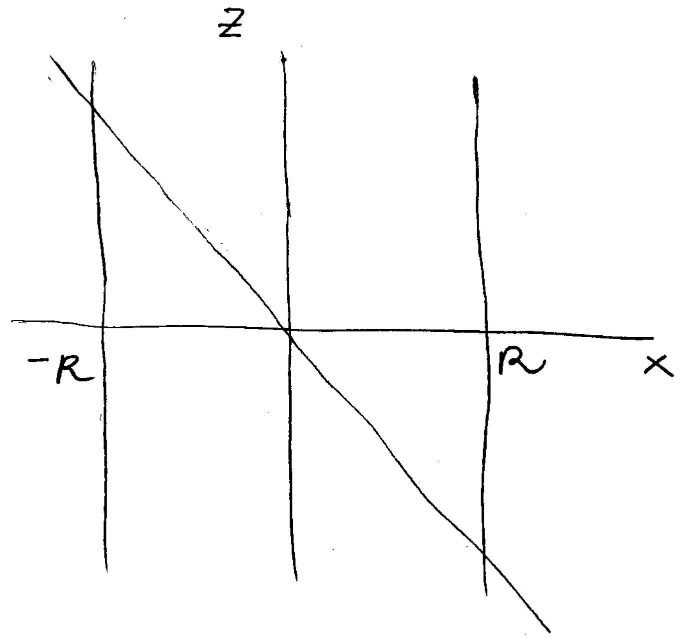
$$\int_S \vec{A}\cdot\vec{n}\,dS = \int_S (\vec{\nabla}\times\vec{b})\cdot\vec{n}\,dS = \oint_C \vec{b}\cdot d\vec{l}$$

gdje C rubna krivulja. Dakle, sveli smo plošni integral na integral po rubnoj krivulji C.

(S) Krivulja

$$x^2 + y^2 = R^2$$

$$x + z = 0$$



Radi se o elipsi u prostoru. Možemo je parametrizirati

$$x = t$$

$$y^2 = R^2 - t^2$$

$$z = -t$$

Za provjeru nam treba

$$\vec{b}\cdot d\vec{l} = (-2xy^2z - xz^3)\cdot dx - x^2y^2\,dz$$

Uvrstimo informaciju o krivulji u ovaj izraz

$$\Big[-2t(R^2-t^2)\cdot(-t) - t\cdot(-t)^3\Big]dt$$
$$-\,t^2\cdot(R^2-t^2)\cdot(-dt)\Big]$$
$$= \Big[2R^2t^2 - 2t^4 + t^4 + R^2t^2 - t^4\Big]dt$$
$$= \Big[3R^2t^2 - 2t^4\Big]dt$$

Parametar t ide u granicama od x, $[-R,R]$

$$\oint_C \vec{b}\cdot d\vec{l} = \int_{-R}^{R}(3R^2t^2-2t^4)\,dt + \int_{R}^{-R}(3R^2t^2-2t^4)\,dt$$
$$= 0$$

U ovom posebnom slučaju, krivulja je takva da integral ne ovisi o putu. U slučaju kružnice u $z=0$ ravnini

$$x = R\cos\phi$$
$$y = R\sin\phi$$
$$z = 0$$

$\vec{b}$ nije konzervativno polje!!

Imamo

$$b_x = b_y = 0$$
$$b_z = -R^2\cos^2\phi\, R^2\sin^2\phi$$

$$\oint_C \vec{b}\cdot d\vec{l} = \int_0^{2\pi} d\phi\, R\cdot\big(R^4\underbrace{\sin^2\phi\cos^2\phi}_{(\frac{1}{2}\sin 2\phi)^2}\big) = \frac{R^5}{4}\int_0^{2\pi} d\phi\,\sin^2 2\phi$$
$$= \frac{R^5}{4}\cdot\frac{1}{2}\cdot 2\pi \neq 0$$

$\boxed{12.1}$

Parametarske jednadžbe sfere radijusa R

$$x = R\sin\theta\cos\varphi$$
$$y = R\sin\theta\sin\varphi$$
$$z = R\cos\theta$$

Normala gleda u radijalnom smjeru. Provjerimo!

$$\frac{\partial\vec{r}}{\partial\theta} = \left(R\cos\theta\cos\varphi,\ R\cos\theta\sin\varphi,\ -R\sin\theta\right)$$

$$\frac{\partial\vec{r}}{\partial\varphi} = \left(-R\sin\theta\sin\varphi,\ R\sin\theta\cos\varphi,\ 0\right)$$

$$\frac{\partial\vec{r}}{\partial\theta}\times\frac{\partial\vec{r}}{\partial\varphi} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ R\cos\theta\cos\varphi & R\cos\theta\sin\varphi & -R\sin\theta \\ -R\sin\theta\sin\varphi & R\sin\theta\cos\varphi & 0 \end{vmatrix} =$$

$$= \vec{e}_x\left(R^2\sin^2\theta\cos\varphi\right) - \vec{e}_y\left(-R^2\sin^2\theta\sin\varphi\right) + \vec{e}_z\left(R^2\sin\theta\cos\theta\cos^2\varphi + R^2\cos\theta\sin\theta\sin^2\varphi\right)$$

$$= \vec{e}_x R^2\sin^2\theta\cos\theta + \vec{e}_y R^2\sin^2\theta\sin\varphi + \vec{e}_z R^2\sin\theta\cos\theta$$

Ovo se može zapisati u obliku

$$\frac{\partial\vec{r}}{\partial\theta}\times\frac{\partial\vec{r}}{\partial\varphi} = R^2\sin\theta\cdot\vec{e}_r$$

Uvrstimo li parametarske jednadžbe sfere u vektorsko polje

$$\vec{A}(\vec{r}) = R^2\vec{r}$$

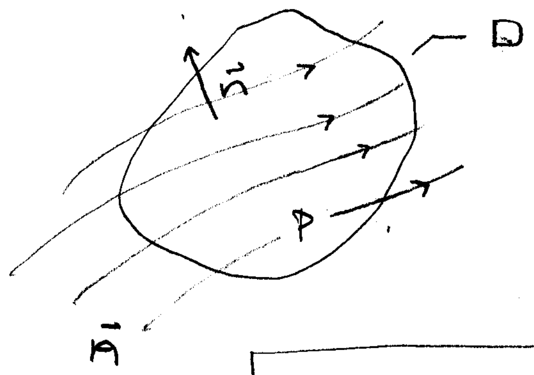
gdje $\vec{r} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z$.

$$\vec{A}\cdot d\vec{S} = \vec{A}\cdot\left(\frac{\partial\vec{r}}{\partial\theta}\times\frac{\partial\vec{r}}{\partial\varphi}\right)d\theta d\varphi = R^2\vec{r}\cdot R\sin\theta\,\vec{r}\,d\theta d\varphi$$

$$= R^3\sin\theta\,\underbrace{\vec{r}\cdot\vec{r}}_{r^2=R^2}\,d\theta d\varphi = R^5\sin\theta\, d\theta d\varphi$$

$$\oint \vec{A}\cdot d\vec{S} = \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta\, R^5 \sin\vartheta = \underline{4\pi R^5}$$

Teorem o divergenciji



D zatvorena ploha koja omeđuje područje P.
$\vec{n}$ je vanjska normala
$\vec{A} = \vec{A}(\vec{r})$ vektorsko polje (neprekidno, parcijalne derivacije neprekidne $\Rightarrow$ klase C^1)

$$\boxed{\oint_D \vec{A}\cdot d\vec{S} = \int_P \vec{\nabla}\cdot\vec{A}\, dV}$$

ZADATAK: treba najprije izračunati $\vec{\nabla}\cdot\vec{A}$

$$\vec{\nabla}\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\frac{\partial A_x}{\partial x} = 3x^2+y^2+z^2; \quad \frac{\partial A_y}{\partial y} = x^2+3y^2+z^2; \quad \frac{\partial A_z}{\partial z} = x^2+y^2+3z^2$$

$$\vec{\nabla}\cdot\vec{A} = 5(\underbrace{x^2+y^2+z^2}_{r^2}) = 5r^2$$

$dV = r^2 \sin\vartheta\, dr\, d\varphi\, d\vartheta$ u sfernim koordinatama (zadatak Z3.6)

$$\int_V \vec{\nabla}\cdot\vec{A}\, dV = \int_0^R \int_0^{2\pi} \int_0^{\pi} 5r^2 \cdot r^2 \sin\vartheta\, dr\, d\vartheta\, d\varphi$$

$$= 5\frac{R^5}{5}\cdot 4\pi = \underline{\underline{4\pi R^5}}$$