(Schaum's Outlines, Complex Variables, str. 138)

Dokažite ***Jensenov*** teorem: pretpostavimo da je $f(z)$ analitička unutar i na kružnici $|z| = R$ osim u točkama b_1 , b_2 , ..., b_m u unutrašnjosti kružnice koje su polovi redova q_1 , q_2 , ..., q_m . Također, u unutrašnjosti kružnice nalaze se i nule za $f(z)$ u točkama a_1 , a_2 , ..., a_n koje su redova p_1 , p_2 , ..., p_n . Ako je $f(0)$ konačna i različita od nule dokažite da je

$$\frac{1}{2\pi}\int_0^{2\pi} \ln\left|f(R\mathrm{e}^{i\theta})\right|\mathrm{d}\theta = \ln\left|f(0)\right| + \sum_{k=1}^{m} p_k \ln\left(\frac{R}{|a_k|}\right) - \sum_{k=1}^{n} q_k \ln\left(\frac{R}{|b_k|}\right) \qquad (1)$$

Dokaz: funkciju $f(z)$ zapišimo u obliku

$$f(z) = g(z)\frac{(z-a_1)^{p_1}(z-a_2)^{p_2}...(z-a_n)^{p_n}}{(z-b_1)^{q_1}(z-b_2)^{q_2}...(z-b_m)^{q_m}} \qquad (2)$$

gdje je $g(z)$ analitička i $g(z) \neq 0$ na $|z| \leq R$. Ako je $g(z)$ analitička, tada je i $\ln|g(z)|$ harmonička na $|z| \leq R$ (dokazano na vježbama!) pa prema teoremu srednje vrijednosti (za realne funkcije) vrijedi

$$\ln|g(0)| = \frac{1}{2\pi}\int_0^{2\pi} \ln|g(R\mathrm{e}^{i\theta})|\,\mathrm{d}\theta \qquad (3)$$

Lijeva strana jednakosti (3), pomoću (2) postaje

$$\ln|f(0)| + \sum_{i=1}^{m} q_i \ln|b_i| - \sum_{j=1}^{n} p_j \ln|a_j| \qquad (4)$$

dok je desna strana

$$\frac{1}{2\pi}\int_0^{2\pi} \ln|f(R\mathrm{e}^{i\theta})|\,\mathrm{d}\theta + \frac{1}{2\pi}\sum_{i=1}^{m} q_i \int_0^{2\pi} \ln|R\mathrm{e}^{i\theta} - b_i|\,\mathrm{d}\theta - \frac{1}{2\pi}\sum_{j=1}^{n} p_j \int_0^{2\pi} \ln|R\mathrm{e}^{i\theta} - a_i|\,\mathrm{d}\theta \qquad (5)$$

Zamijenimo varijablu integracije u članovima koji se nalaze unutar suma u (5). Imamo

$$\begin{aligned} R\mathrm{e}^{i\theta} - b_i &= R\mathrm{e}^{i\varphi} \\ R\mathrm{e}^{i\theta} - a_i &= R\mathrm{e}^{i\varphi} \end{aligned} \qquad (6)$$

i

$$iR\mathrm{e}^{i\theta}\,\mathrm{d}\theta = iR\mathrm{e}^{i\varphi}\,\mathrm{d}\varphi \qquad (7)$$

Na primjer,

$$\begin{aligned} \int_0^{2\pi} \ln|R\mathrm{e}^{i\theta} - b_i|\,\mathrm{e}^{-i\theta}\mathrm{e}^{i\theta}\,\mathrm{d}\theta &= \int_0^{2\pi} \ln|R\mathrm{e}^{i\varphi}|\left(\mathrm{e}^{-i\varphi} + \frac{b_i^*}{R}\right)\mathrm{e}^{i\varphi}\,\mathrm{d}\varphi \\ &= \ln|R|\int_0^{2\pi}\mathrm{d}\varphi + \frac{b_i^*\ln|R|}{R}\underbrace{\int_0^{2\pi}\mathrm{e}^{i\varphi}\,\mathrm{d}\varphi}_{=0} = 2\pi\ln|R| \end{aligned} \qquad (8)$$

Izraz (5) postaje

$$\frac{1}{2\pi}\int_0^{2\pi} \ln|f(R\mathrm{e}^{i\theta})|\,\mathrm{d}\theta + \sum_{i=1}^{m} q_i \ln|R| - \sum_{j=1}^{n} p_j \ln|R| \qquad (9)$$

Jednakost (3) kojoj smo izračunali lijevu stranu (4) i desnu (9) postaje

$$\ln|f(0)| + \sum_{i=1}^{m} q_i \ln|b_i| - \sum_{j=1}^{n} p_j \ln|a_j| = \frac{1}{2\pi}\int_0^{2\pi} \ln|f(R\mathrm{e}^{i\theta})|\,\mathrm{d}\theta + \sum_{i=1}^{m} q_i \ln|R| - \sum_{j=1}^{n} p_j \ln|R| \qquad (10)$$

pa je teorem dokazan.