# MATEMATIČKE METODE FIZIKE II

## Prvi kolokvij 22.05.2014.

**1.** Odredite diferencijalnu jednadžbu svih kružnica u ravnini *xy*.

**2.** Riješite jednadžbu

$$(x+y^2)\,\mathrm{d}x - 2xy\,\mathrm{d}y = 0$$

***Uputa*:** Eulerov multiplikator je funkcija $\mu = \mu(x)$.

**3.** Upotrijebite metodu varijacije konstanti da riješite jednadžbu:

$$y'' + 4y = 2\,\mathrm{tg}\,x$$

**4.** Riješite sustav jednadžbi:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = 2t(x_1^2 + x_2^2)$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = 4tx_1x_2$$

**5.** Koja veza mora postojati između funkcija $f(x)$ i $\varphi(x)$ da bi jednadžba

$$y'' + f(x)y' + \phi(x)y = 0$$

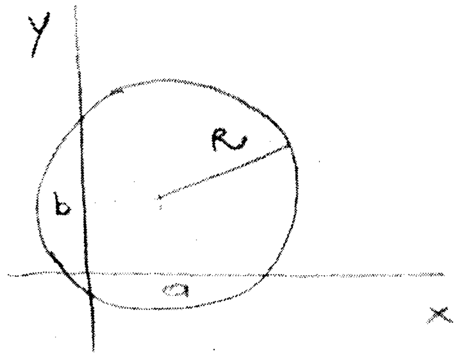
imala dva partikularna rješenja od kojih je jedno kvadrat drugoga?

**1.**

Kružnice u ravnini imaju jednadžbu

$$(x-a)^2+(y-b)^2=R^2$$

gdje su $a, b, R$ parametri. Uređeni par $(a,b)$ su koordinate središta kružnice, a $R$ je polumjer kružnice.



Deriviramo gornju jednadžbu

$$2(x-a)+2(y-b)y'=0$$

Deriviramo još jednom

$$1+y'^2+(y-b)y''=0$$

$$y-b=-\frac{1+y'^2}{y''}$$

Deriviramo još jednom

$$2y'y''+y'y''+(y-b)y'''=0$$

$$3y'y''-\frac{1+y'^2}{y''}y'''=0$$

$$\boxed{3y'y''^2-(1+y'^2)y'''=0}$$

Može se pokazati da je ova jednadžba ekvivalentna jednadžbi

$$\frac{d}{dx}\left[\frac{(1+y'^2)^{3/2}}{y''}\right]=0$$

$\boxed{2.}$

$$(x+y^2)\,dx - 2xy\,dy = 0$$

$$P(x,y) = x + y^2$$

$$Q(x,y) = -2xy$$

$$\frac{\partial P}{\partial y} = 2y$$

$$\frac{\partial Q}{\partial x} = -2y$$

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x} \quad \text{nije egzaktna}$$

Budući da $\mu = \mu(x)$ imamo

$$\frac{\mu'}{\mu} = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{1}{-2xy}\left(2y-(-2y)\right)$$

$$= -\frac{2}{x}$$

$$\ln|\mu| = -2\ln|x| + C$$

$$\mu = x^{-2}$$

$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy = 0$$

$$= x^{-2}(x+y^2)dx - 2\frac{y}{x}dy = 0$$

$$\frac{\partial U}{\partial x} = x^{-1} + x^{-2}y^2 \quad / \int$$

$$U = \ln|x| - x^{-1}y^2 + f(y)$$

$$\frac{\partial U}{\partial y} = -2x^{-1}y + \frac{df}{dy} = -2x^{-1}y$$

$$\frac{df}{dy} = 0 \Rightarrow f = C_1$$

$$\ln|x| - x^{-1}y^2 + c_1 = \text{konst.}$$

Imamo,

$$\boxed{\ln|x| - x^{-1}y^2 = c_2}$$

ili,

$$\ln|x| = c_2 + x^{-1}y^2$$

$$x = \exp(c_2 + x^{-1}y^2)$$

$$= c_3\, e^{y^2/x}$$

$$\boxed{x = c_3\, e^{y^2/x}}$$

$\boxed{3.}$

$$y'' + 4y = 2\,tg\,x$$

Riješimo najprije homogenu jednadžbu.

$$y'' + 4y = 0$$

Karakteristična jednadžba: $r^2 + 4 = 0$

$$r_{1,2} = \pm 2i$$

Rješenje: $y_H = c_1 \sin 2x + c_2 \cos 2x$

Rješenje nehomogene jednadžbe tražimo u obliku

$$y = c_1(x) \sin 2x + c_2(x) \cos 2x$$

Jednadžbe:

$$c_1' \sin 2x + c_2' \cos 2x = 0$$

$$2c_1' \cos 2x + 2c_2'(-\sin 2x) = 2\,tg\,x$$

Iz prve: $$c_2' = -\frac{c_1' \sin 2x}{\cos 2x}$$

$$c_1' \cos 2x + \frac{\sin^2 2x}{\cos 2x} c_1' = tg\,x \;/\cos 2x$$

$$c_1' = tg\,x \cdot \cos 2x$$

Uvrstimo u $c_2'$

$$c_2' = -tg\,x \cdot 2 \sin x \cos x = -2 \sin^2 x$$

$$c_2(x) = -2 \cdot \left(\frac{1}{2}x - \frac{1}{4}\sin 2x\right) + C_3$$

$$= -x + \frac{1}{2}\sin 2x + C_3$$

$$c_1'(x) = \frac{\sin x}{\cos x} \cdot (2\cos^2 x - 1)$$

$\cos x = u$

$-\sin x\, dx = du$

$$dc_1(x) = -du\,\frac{2u^2-1}{u} = \frac{du}{u} - 2u\,du$$

$$c_1(x) = \ln|u| - u^2 + c_4$$

$$= \ln|\cos x| - \cos^2 x + c_4$$

Rješenje nehomogene jednadžbe

$$y = (\ln|\cos x| \underbrace{-\cos^2 x + c_4}_{\frac{1}{2}\cos 2x + \frac{1}{2}})\sin 2x + (-x + \tfrac{1}{2}\sin 2x + c_3)\cos 2x$$

$$c_4 + \frac{1}{2} \to c_5$$

$$= c_5 \sin 2x + c_3 \cos 2x + \sin 2x \ln|\cos x| - x\cos 2x$$

**4.**

Ponekad je zgodno sustave diferencijalnih jednadžbi rješavati metodom pronalaženja integrabilnih kombinacija. Jednadžbe sustava se zbrajaju, oduzimaju, množe ili dijele da se dobiju pogodne kombinacije naših funkcija koje se onda mogu integrirati.

ZADATAK:

$$\frac{dx_1}{dt} = 2t(x_1^2 + x_2^2)$$

$$\frac{dx_2}{dt} = 4t x_1 x_2$$

Zbrojimo ove jednadžbe

$$\frac{d}{dt}(x_1 + x_2) = 2t(x_1 + x_2)^2$$

te dobivenu jednadžbu napišimo u obliku

$$\frac{d(x_1 + x_2)}{(x_1 + x_2)^2} = 2t\,dt$$

Integriramo

$$-\frac{1}{x_1 + x_2} = t^2 + C_1 \qquad (1)$$

Ako jednadžbe oduzmemo

$$\frac{d}{dt}(x_1 - x_2) = 2t(x_1 - x_2)^2$$

$$\frac{d(x_1 - x_2)}{(x_1 - x_2)^2} = 2t\,dt$$

Integriramo

$$-\frac{1}{(x_1 - x_2)} = t^2 + C_2 \qquad (2)$$

Jednadžbe (1) i (2) napišimo u obliku

$$x_1 + x_2 = -\frac{1}{C_1 + t^2}$$

Ako ove jednadžbe zbrojimo pa oduzmemo, dobit ćemo rješenja

$$x_1 = -\frac{1}{2}\left(\frac{1}{c_1+t^2}+\frac{1}{c_2+t^2}\right)$$

$$x_2 = -\frac{1}{2}\left(\frac{1}{c_1+t^2}-\frac{1}{c_2+t^2}\right)$$

**5.**

$$y_1'' + f(x)\,y_1' + \varphi(x)\,y_1 = 0 \qquad (*)$$

$$y_2'' + f(x)\,y_2' + \varphi(x)\,y_2 = 0$$

$$y_2 = y_1^2$$

$$y_2' = 2y_1y_1'$$

$$y_2'' = 2y_1'^2 + 2y_1y_1''$$

$$2y_1'^2 + \underline{2y_1y_1''} + \underline{f(x)\cdot 2y_1y_1'} + \varphi(x)\,y_1^2 = 0$$

$$2y_1'^2 + 2y_1\big(\underbrace{y_1'' + f(x)\,y_1'}_{-\varphi(x)y_1}\big) + \varphi(x)\,y_1^2 = 0$$

$$2y_1'^2 - \underline{2y_1^2\varphi(x)} + \underline{\varphi(x)\,y_1^2} = 0$$

$$\frac{y_1'}{y_1} = \pm\frac{1}{\sqrt2}\sqrt{\varphi(x)}$$

$$\ln|y_1| = \pm\frac{1}{\sqrt2}\int\sqrt{\varphi(x)}\,dx + C$$

$$y_1 = C_1 e^{\pm\frac{1}{\sqrt2}\int\sqrt{\varphi(x)}\,dx}$$

Uvrstimo u jednadžbu $(*)$

$$y_1' = C_1 e^{\pm\frac{1}{\sqrt2}\int\sqrt{\varphi(x)}\,dx}\cdot(\pm)\cdot\frac{1}{\sqrt2}\sqrt{\varphi(x)}$$

$$y_1'' = C_1 e^{\pm\frac{1}{\sqrt2}\int\sqrt{\varphi(x)}\,dx}\cdot\frac{1}{2}\varphi(x) \pm C_1 e^{\pm\frac{1}{\sqrt2}\int\sqrt{\varphi(x)}\,dx}\cdot\frac{1}{2\sqrt2}\cdot\frac{\varphi'}{\sqrt\varphi}$$

$$\frac{1}{2}\varphi(x) \pm \frac{1}{2\sqrt{2}}\frac{\varphi'}{\sqrt{\varphi}} \pm f(x)\frac{1}{\sqrt{2}}\sqrt{\varphi(x)} + \varphi(x) = 0$$

$$\frac{3}{2}\varphi(x)\sqrt{\varphi(x)} \pm \frac{1}{2\sqrt{2}}\varphi' \pm f(x)\varphi(x)\cdot\frac{1}{\sqrt{2}} = 0 \quad /\cdot 2\sqrt{2}$$

ili

$$\varphi' \pm 3\varphi\sqrt{2\varphi} + 2f(x)\varphi(x) = 0$$