NAPREDNA ELEKTRODINAMIKA

Prvi kolokvij 2. 11. 2023.

ZADATAK 1 Sferne ljuska sastoji se od dvije koncetrične i vodljive sfere na polumjerima a i b ($a < b$). Sfera polumjera a je na potencijalu

$$\Phi(a,\theta,\phi) = V(\theta,\phi) = V_0\left(\cos\phi\sin\theta + \cos\theta\right)$$

Sfera polumjera b je na potencijalu jednakom nuli. Upotrijebite Greenovu funkciju za sfernu ljusku da nađete potencijal između sfera.

***Uputa*:** Greenova funkcija za sfernu ljusku omeđenu sferama $r = a$ i $r = b$ glasi

$$G(\mathbf{r},\mathbf{r}') = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{Y_{lm}^{*}(\theta',\phi')Y_{lm}(\theta,\phi)}{(2l+1)\left[1-(a/b)^{2l+1}\right]}\left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right)$$

Izrazite $\cos\phi\cos\theta$ i $\cos\theta$ u potencijalu $V(\theta,\phi)$ pomoću sfernih harmonika.

ZADATAK 2 (a) Počevši od izraza za Lorentzovu silu

$$\frac{d\mathbf{P}_{\text{mech}}}{dt} = \int_V \left(\rho\mathbf{E} + \mathbf{J}\times\mathbf{B}\right)d^3r$$

gdje je $\mathbf{P}_{mech}$ ukupni impuls čestica u volumenu V, pokažite da je u dipolnoj aproksimaciji sila koja djeluje na neutralni atom u mirovanju jednaka

$$\frac{d\mathbf{P}_{\text{mech}}}{dt} = \left(\mathbf{d}\cdot\nabla\right)\mathbf{E} + \dot{\mathbf{d}}\times\mathbf{B}$$

gdje je $\mathbf{d}$ atomski dipolni moment, a $\mathbf{E}$ i $\mathbf{B}$ su električno i magnetsko polje na položaju atoma.

(b) Za ravni val frekvencije ω koji se širi u nemagnetskom dielektriku niske gustoće s indeksom loma $n(\omega)$, pokažite da je brzina promjene mehaničkog impulsa po volumenu $\mathbf{g}_{\text{mech}}$ povezana s elektromagnetskim impulsom $\mathbf{g}_{\text{em}}$ ravnog vala kroz relaciju

$$\frac{d\mathbf{g}_{\text{mech}}}{dt} = \frac{1}{2}\left(n^2 - 1\right)\frac{d\mathbf{g}_{\text{em}}}{dt}$$

gdje je $\mathbf{g}_{\text{em}} = (1/c^2)\mathbf{E}\times\mathbf{B}$.

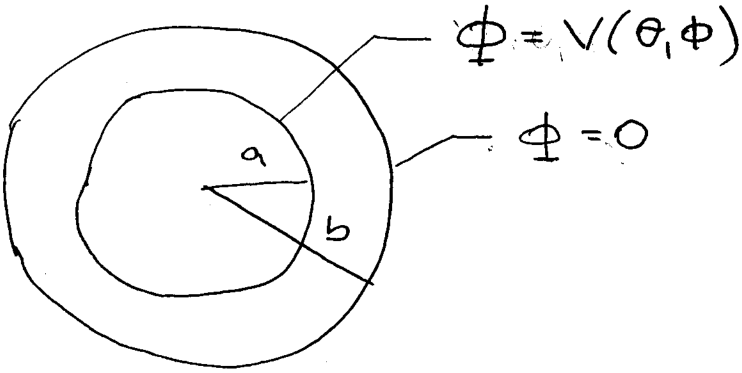
***Uputa*:** pod (a), koristite izraz za efektivnu gustoću naboja koja odgovara dipolu momenta $\mathbf{p}(t)$ na položaju $\mathbf{r}_0$

$$\rho_{\text{eff}}(\mathbf{r}) = -\mathbf{p}\cdot\nabla\delta(\mathbf{r}-\mathbf{r}_0)$$

te efektivnu gustoću struje

$$\mathbf{J}(\mathbf{r}) = \dot{\mathbf{p}}\,\delta(\mathbf{r}-\mathbf{r}_0)$$

1.



Potencijal računamo po formuli

$$\Phi(\vec r) = -\frac{1}{4\pi}\oint_S \Phi(\vec r\,')\,\frac{\partial G}{\partial n'}\,dS'$$

Po sferi $r=b$ potencijal je nula pa ovaj integral postaje

$$\Phi(\vec r) = -\frac{1}{4\pi}\int_{r'=a} \Phi(\vec r\,')\,\frac{\partial G}{\partial n'}\,dS'$$

Normala (vanjska) na plohu $r=a$ je $-\vec e_r$ pa

$$\frac{\partial G}{\partial n'} = \vec\nabla' G\cdot\vec n\,' = -\frac{\partial G}{\partial r'}$$

U Greenovoj je funkciji $r_< = r'$, a $r_> = r$. Imamo

$$\frac{\partial}{\partial r'}\left[4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{Y_{lm}^*(\theta',\phi')\,Y_{lm}(\theta,\phi)}{(2l+1)\left[1-(a/b)^{2l+1}\right]}\cdot\left(r'^{\,l}-\frac{a^{2l+1}}{r'^{\,l+1}}\right)\cdot\left(\frac{1}{r^{l+1}}-\frac{r^l}{b^{2l+1}}\right)\right.$$

$$= 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{Y_{lm}^*\,Y_{lm}}{(2l+1)\left[1-(a/b)^{2l+1}\right]}\cdot\left(l\,r'^{\,l-1}+(l+1)\frac{a^{2l+1}}{r'^{\,l+2}}\right)\cdot\left(\frac{1}{r^{l+1}}-\frac{r^l}{b^{2l+1}}\right)$$

Potencijal $V(\theta,\phi)$ treba napisati pomoću sfernih harmonika.

$$Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{i\phi}$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta$$

Također je

$$Y_{1,-1} = (-1)\, Y_{11}^{*} = +\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{-i\phi}$$

Imamo,

$$\cos\theta = \sqrt{\frac{4\pi}{3}}\, Y_{10}$$

te

$$\cos\phi \sin\theta = \frac{1}{2}\left(e^{i\phi} + e^{-i\phi}\right)\sin\theta$$

$$= \frac{1}{2}\cdot\left(-\sqrt{\frac{8\pi}{3}}\, Y_{11} + \sqrt{\frac{8\pi}{3}}\, Y_{1,-1}\right)$$

Potencijal $V(\theta,\phi)$ je

$$V(\theta,\phi) = V_0\cdot\sqrt{\frac{4\pi}{3}}\, Y_{10} + V_0\cdot\frac{1}{2}\sqrt{\frac{8\pi}{3}}\left(Y_{1,-1} - Y_{11}\right)$$

Uvrstimo sve međurezultate u izraz za potencijal

$$\Phi(\vec{r}) = \left(+\frac{1}{4\pi}\right)\cdot 4\pi \sum_{l,m} \frac{Y_{lm}(\theta,\phi)}{1-(a/b)^{2l+1}}\cdot a^{l-1}\cdot\left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right)$$

$$\cdot a^2\cdot V_0\left[\sqrt{\frac{4\pi}{3}}\int Y_{lm}^{*}\, Y_{10}\, d\Omega' + \frac{1}{2}\sqrt{\frac{8\pi}{3}}\int Y_{lm}^{*}\, Y_{1,-1}\, d\Omega'\right.$$

$$\left. - \frac{1}{2}\sqrt{\frac{8\pi}{3}}\int Y_{lm}^{*}\, Y_{11}\, d\Omega'\right]$$

Integrali su, redom

$$\int Y_{lm}^{*}\, Y_{10}\, d\Omega' = \delta_{l1}\,\delta_{m0}$$

$$\int Y_{lm}^{*} Y_{1,-1}\, d\Omega' = \delta_{l1}\, \delta_{m,-1}$$

$$\int Y_{lm}^{*} Y_{11}\, d\Omega' = \delta_{l1}\, \delta_{m1}$$

Potencijal poretaje

$$\Phi(\vec{r}) = + V_0 \cdot \frac{a^2}{1-(a/b)^3} \left(\frac{1}{r^2} - \frac{r}{b^3} \right) \cdot \left[\sqrt{\frac{4\pi}{3}}\, Y_{10} \right.$$

$$\left. + \frac{1}{2}\sqrt{\frac{8\pi}{3}}\, Y_{1,-1} - \frac{1}{2}\sqrt{\frac{8\pi}{3}}\, Y_{11} \right]$$

$$= + \frac{V_0}{1-(a/b)^3} \cdot \left(\left(\frac{a}{r}\right)^2 - \frac{a^2 r}{b^3} \right) \cdot \cos\phi \sin\theta$$

$\boxed{2.}$

(a) Za atomski dipolni moment $\vec{d}$ na položaju $\vec{r}_0$

$$\rho_{eff} = -\vec{d}\cdot\vec{\nabla}\delta(\vec{r}-\vec{r}_0)$$
$$\vec{J} = \dot{\vec{d}}\,\delta(\vec{r}-\vec{r}_0)\ ; \quad \dot{\vec{d}} = \frac{d}{dt}(\vec{d})$$

$$\int \rho_{eff}\vec{E}\,d^3r = -\int \vec{d}\cdot\vec{\nabla}\delta(\vec{r}-\vec{r}_0)\,\vec{E}(\vec{r})\,d^3r$$
$$= (\vec{d}\cdot\vec{\nabla})\vec{E}(\vec{r}_0) \qquad (*)$$

$$\int \vec{J}_{eff}\times\vec{B}\,d^3r = \int \dot{\vec{d}}\times\vec{B}(\vec{r})\,\delta(\vec{r}-\vec{r}_0)\,d^3r$$
$$= \dot{\vec{d}}\times\vec{B}(\vec{r}_0)$$

Dokažimo (*). Imamo za E_x

$$\int \rho_{eff}E_x\,d^3r = -\vec{d}\cdot\int \vec{\nabla}\delta(\vec{r}-\vec{r}_0)\,E_x\,d^3r$$

$$\vec{\nabla}\left[\delta(\vec{r}-\vec{r}_0)E_x\right] = \vec{\nabla}\delta(\vec{r}-\vec{r}_0)E_x + \delta(\vec{r}-\vec{r}_0)\vec{\nabla}E_x$$

Integral

$$\int \vec{\nabla}\left[\delta(\vec{r}-\vec{r}_0)E_x\right]d^3r = \oint_{S\to\infty} \delta(\vec{r}-\vec{r}_0)E_x\,\vec{n}\,dS \to 0$$

jer točka $\vec{r}_0$ ne leži na beskonačnoj plohi. Prema tome,

$$\int \rho_{eff}E_x\,d^3r = +\vec{d}\cdot\int \vec{\nabla}E_x\,\delta(\vec{r}-\vec{r}_0)\,d^3r$$
$$= (\vec{d}\cdot\vec{\nabla})E_x(\vec{r}_0)$$

Slično vrijedi i za E_y i E_z komponente.

(b) Integriramo li po volumenu dielektrika, ukupni mehanički impuls za sredstvo postaje

$$\int \left[(\vec{P}\cdot\vec{\nabla})\vec{E} + \dot{\vec{P}} \times \vec{B} \right] dV \qquad (*)$$

gdje je $\vec{P}$ polarizacija sredstva. Polja $\vec{E}$ i $\vec{B}$ su makroskopska polja jer dielektrik ima malu gustoću (ne treba računati s lokalnim poljem!). Veza između polarizacije i makroskopskog polja glasi

$$\vec{P} = (\epsilon - \epsilon_0)\vec{E}$$

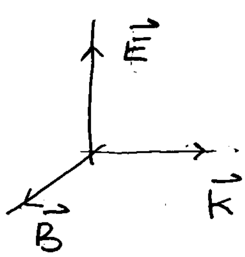
pa je integral (*)

$$(\epsilon - \epsilon_0) \int \left[(\vec{E}\cdot\vec{\nabla})\vec{E} + \dot{\vec{E}} \times \vec{B} \right] dV$$

Iz ove relacije je derivacija po vremenu ukupne gustoće mehaničkog impulsa jednaka

$$\frac{d\vec{g}_{mech}}{dt} = (\epsilon - \epsilon_0)\left[(\vec{E}\cdot\vec{\nabla})\vec{E} + \dot{\vec{E}} \times \vec{B} \right]$$

Odaberimo koordinatni sustav tako da upadni val ima sljedeće komponente



$$\vec{E}_i = E_0\,\vec{e}_x\, e^{i(kz-\omega t)}$$

$$\vec{B}_i = B_0\,\vec{e}_y\, e^{i(kz-\omega t)}$$

$$\vec{k} = k\,\vec{e}_z$$

Tada i makroskopsko polje ima iste komponente.

Računamo

$$(\vec{E}\cdot\vec{\nabla})\vec{E} = E_x \frac{\partial}{\partial x}\vec{E} = 0 \quad \text{jer } \vec{E} \text{ ovisi samo o } z$$

$$\dot{\vec{E}} \times \vec{B} = \det\begin{pmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \dot{E}_x & 0 & 0 \\ 0 & B_y & 0 \end{pmatrix} = \dot{E}_x B_y \vec{e}_z$$

$$\left.\begin{aligned} \dot{E}_x B_y &= E_0 B_0 \cdot (-i\omega t) \\ E_x \dot{B}_y &= E_0 B_0 (-i\omega t) \end{aligned}\right\} \quad \dot{E}_x B_y = E_x \dot{B}_y$$

Prema tome,

$$\dot{E}_x B_y = \frac{1}{2}\frac{d}{dt}(E_x B_y)$$

i

$$(\dot{\vec{E}} \times \vec{B})_z = \frac{1}{2}\frac{d}{dt}(E_x B_y)$$

Vrijedi

$$n^2 = \underset{\text{relativna permitivnost}}{\epsilon_r}\mu_r \simeq \frac{\epsilon}{\epsilon_0} \quad \text{jer je } \mu_r \simeq 1$$

za nemagnetna sredstva.

Imamo

$$\epsilon - \epsilon_0 = \epsilon_0 n^2 - \epsilon_0 = \epsilon_0(n^2-1)$$

$$H = \frac{B}{\mu_0}$$

pa dobivamo za vektorsku jednadžbu $\vec{g}_{eM}$

$$\frac{d\vec{g}_{mech}}{dt} = \frac{\mu_0\epsilon_0(n^2-1)}{2}\frac{d}{dt}(\vec{E}\times\vec{H}) = \frac{1}{2c^2}(n^2-1)\frac{d\vec{g}_{em}}{dt}$$