**SVEUČILIŠTE U RIJECI, FAKULTET ZA FIZIKU**

***Diplomski studij Fizika*** **Ak. god. 2023./2024.**

# NAPREDNA ELEKTRODINAMIKA

## Drugi kolokvij 7. 12. 2023.

ZADATAK 1 Kao posljedicu zakona očuvanja impulsa za sustav naboja i polja, može se izvesti formula za silu na volumen $V$

$$\mathbf{F} = \int_V \left( \nabla \cdot \overleftrightarrow{\mathbf{T}} - \frac{1}{c^2} \frac{\partial \mathbf{S}}{\partial t} \right) d^3 r$$

Upotrijebit ćemo ovu formulu da pronađemo vremenski prosjek sile na svaku stranicu rezonantne šupljine oblika kvadra. Šupljina je definirana nejednakostima $0 \le x \le a$, $0 \le y \le a$ i $0 \le z \le h$, a stranice šupljine su savršeni vodiči. Pretpostavite da unutar šupljine polje titra u pobuđenom modu u kojem je

$$E_x = E_y = B_z = 0$$

$$E_z = E_0 \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{a}\right) e^{-i\omega t}$$

(a) Uvrstite polje u valnu jednadžbu za polje **E** i nađite frekvenciju $\omega$ ovog moda.
(b) Pomoću Faradayevog zakona, pronađite magnetsko polje **B**.
(c) Kakve su sile na stranice $x = 0$ i $x = a$, na primjer, ako ukupna sila na šupljinu mora biti jednaka nuli?
(d) Pokažite da je vremenski prosjek derivacije Poyntingova vektora $\langle \partial \mathbf{S} / \partial t \rangle = 0$ te transformirajte izraz za vremenski prosjek sile pomoću teorema o divergenciji u plošni integral.
(e) Iskoristite rezultat pod (d) i pronađite prosjek sile na stranice $x = 0$, $y = 0$ i $z = 0$.

ZADATAK 2 Počevši od izraza za ukupnu energiju proizvoljne superpozicije ravnih elektromagnetskih valova u vakuumu, pokažite da je ukupan broj fotona $N$ dan integralom

$$N = \frac{\epsilon_0}{4\pi^2 \hbar c} \int_V d^3 r \int_V d^3 r' \left[ \frac{\mathbf{E}(\mathbf{r},t) \cdot \mathbf{E}(\mathbf{r}',t) + c^2 \mathbf{B}(\mathbf{r},t) \cdot \mathbf{B}(\mathbf{r}',t)}{|\mathbf{r} - \mathbf{r}'|^2} \right]$$

Broj fotona za svaki ravni val valnog vektora **k** i polarizacije $\mathbf{e}_\lambda$ definiran je kao energija tog vala podijeljena s $\hbar ck$.
***Uputa*:** izrazi za ravne valove dani su formulama 7.8 i 7.11 u knjizi od Jacksona

$$\mathbf{E}(\mathbf{r},t) = \boldsymbol{\mathcal{E}} e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}$$

$$\mathbf{B}(\mathbf{r},t) = \boldsymbol{\mathcal{B}} e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}$$

$$\boldsymbol{\mathcal{B}} = \sqrt{\mu\epsilon}\, \mathbf{n} \times \boldsymbol{\mathcal{E}}$$

gdje je $\mathbf{k} = k\mathbf{n}$, a gustoća energije EM polja glasi

$$u = \frac{1}{2}\left( \epsilon_0 \mathbf{E} \cdot \mathbf{E} + \frac{1}{\mu_0} \mathbf{B} \cdot \mathbf{B} \right)$$

$\boxed{1.}$

(a) Uvrstimo polje u valnu jednadžbu

$$\vec{\nabla}^2 E_z = \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2}$$

$$\vec{\nabla}^2 E_z = \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} = E_0 \cdot \left(-\frac{\pi^2}{a^2}\right) \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{a}\right) e^{-i\omega t}$$

$$+ E_0\left(-\frac{\pi^2}{a^2}\right) \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{a}\right) e^{-i\omega t}$$

$$\frac{\partial^2 E_z}{\partial t^2} = -\omega^2 E_0 \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{a}\right) e^{-i\omega t}$$

Imamo

$$-\frac{2\pi^2}{a^2} = -\frac{\omega^2}{c^2} \Rightarrow \boxed{\omega = \frac{c\pi}{a}\sqrt{2}}$$

(b) Zbog $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = i\omega \vec{B}$, gdje pretpostavljamo

$$\vec{B} \propto e^{-i\omega t}$$

$$\vec{\nabla} \times \vec{E} = \det\begin{pmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \partial_x & \partial_y & \partial_z \\ E_x & E_y & E_z \end{pmatrix} = \vec{e}_x \frac{\partial E_z}{\partial y} - \vec{e}_y \frac{\partial E_z}{\partial x}$$

$$\frac{\partial E_z}{\partial y} = E_0 \frac{\pi}{a} \sin\left(\frac{\pi}{a}x\right) \cos\left(\frac{\pi}{a}y\right) e^{-i\omega t}$$

$$\frac{\partial E_z}{\partial x} = E_0 \frac{\pi}{a} \cos\left(\frac{\pi}{a}x\right) \sin\left(\frac{\pi}{a}y\right) e^{-i\omega t}$$

Polje $\vec{B}$ glasi

$$\vec{B} = \frac{\pi E_0}{i\omega a} \sin\left(\frac{\pi}{a}x\right) \cos\left(\frac{\pi}{a}y\right) e^{-i\omega t} \vec{e}_x$$
$$- \frac{\pi E_0}{i\omega a} \cos\left(\frac{\pi}{a}x\right) \sin\left(\frac{\pi}{a}y\right) e^{-i\omega t} \vec{e}_y$$

(c) Ukupna sila na šupljinu mora biti jednaka nuli jer šupljina generira polje. Zato sile na suprotne stranice kvadra moraju biti jednake po iznosu, ali suprotnog smjera.

(d)
$$\vec{S} = (\mathrm{Re}\,\vec{E}) \times (\mathrm{Re}\,\vec{B}) \cdot \frac{1}{\mu_0}$$
$$= \vec{E}(\vec{r}) \cos\omega t \times \vec{B}(\vec{r}) \sin\omega t \cdot \frac{1}{\mu_0} = \frac{1}{2\mu_0} \vec{E}(\vec{r}) \times \vec{B}(\vec{r}) \sin 2\omega t$$

jer je $\mathrm{Re}\, e^{-i\omega t} = \cos\omega t$ i $\mathrm{Re}\, \frac{1}{i} e^{-i\omega t} = \sin\omega t$

$$\frac{\partial \vec{S}}{\partial t} = \vec{E}(\vec{r}) \times \vec{B}(\vec{r}) \cdot \frac{1}{2\mu_0} \cdot \cos(2\omega t) \cdot 2\omega$$

$$\left\langle \frac{\partial \vec{S}}{\partial t} \right\rangle \propto \frac{1}{T} \int_0^T \cos(2\omega t)\, dt = \frac{1}{T} \cdot \left. \frac{\sin 2\omega t}{2\omega} \right|_0^T$$
$$= \frac{1}{4\pi} \cdot \left[ \sin 4\pi - \sin 0 \right] = 0$$

gdje je $T = \frac{2\pi}{\omega}$. Vremenski prosjek sile

$$\langle \vec{F} \rangle = \left\langle \int_V \vec{\nabla} \cdot \overleftrightarrow{T}\, d^3 r \right\rangle = \left\langle \oint_S \overleftrightarrow{T} \cdot \vec{n}\, dS \right\rangle$$

gdje je $S$ ploha koja omeđuje unutrašnjost šupljine volumena $V$. Ovaj izraz možemo i zapisati pomoću komponenti. Za $i$-tu komponentu sile imamo

$$\int_V \frac{\partial T_{ij}}{\partial x_j}\, dV = \oint_S \sum_j T_{ij}\, n_j\, dS$$

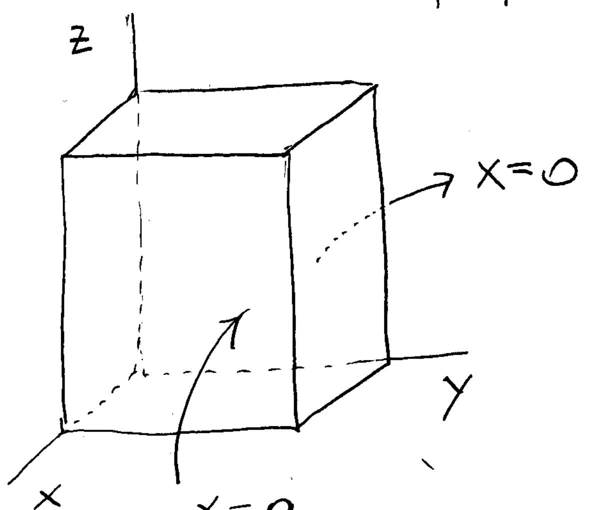
Na primjer, za $i=1$ imamo vektorsku funkciju

$$\vec{a} = \left( T_{11}, T_{12}, T_{13} \right)$$

$$\vec{\nabla} \cdot \vec{a} = \frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3}$$

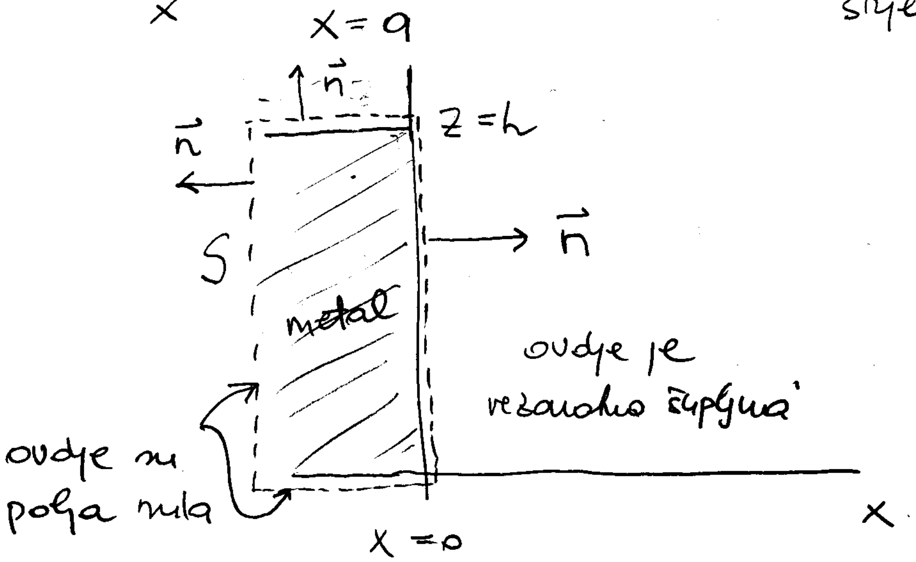
$$\vec{a}_1 \cdot \vec{n} = T_{11} n_1 + T_{12} n_2 + T_{13} n_3$$

(e) Tražimo vremenski prosjek sile na stranicu $x=0$.




Odabir plohe S za izračun sila na stranicu $x=0$: to je kvadar kojem se jedna stranica poklapa s $x=0$ i koji sadrži $x=0$ stijenku rezonantne šupljine

$$\vec{n} = \vec{e}_x$$




Maxwellov tenzor naprezanja

$$T_{ij} = \epsilon_0 \left[ E_i E_j + c^2 B_i B_j - \frac{1}{2} \left( \vec{E} \cdot \vec{E} + c^2 \vec{B} \cdot \vec{B} \right) \delta_{ij} \right]$$

Po $x=0$ vanjska normala za $x=0$ stijenku je $\vec{n} = \vec{e}_x$, odnosno, $n_1 = 1$, $n_2 = n_3 = 0$

Imamo

$$\sum_j T_{ij} n_j = T_{i1} n_1 = T_{i1} = \epsilon_0 \Big[ E_i E_1 + c^2 B_i B_1 - \frac{1}{2} \left( \vec{E} \cdot \vec{E} + c^2 \vec{B} \cdot \vec{B} \right) \delta_{i1} \Big]$$

Računamo matrične elemente za $\overleftrightarrow{T}$; $E_1 = E_2 = 0$; $B_3 = 0$

$$T_{11} = \epsilon_0 \left[ c^2 B_1^2 - \frac{1}{2} \left( E_3^2 + c^2 (B_1^2 + B_2^2) \right) \right]$$
$$= \epsilon_0 \left[ -\frac{1}{2} E_3^2 + \frac{1}{2} c^2 B_1^2 - \frac{1}{2} c^2 B_2^2 \right]$$

$$T_{21} = \epsilon_0 B_2 B_1 c^2$$

$$T_{31} = \epsilon_0 c^2 B_3 B_1 = 0$$

Na $x=0$ je:

$$B_1(x=0) = 0$$
$$B_2(x=0) = \frac{\pi E_0}{\omega a} \sin\left(\frac{\pi}{a} y\right) \sin \omega t$$
$$E_3(x=0) = 0$$

Prema tome:

$$T_{11} = \epsilon_0 \cdot \left(-\frac{1}{2}\right) c^2 \left(\frac{\pi E_0}{\omega a}\right)^2 \sin^2\left(\frac{\pi}{a} y\right) \sin^2 \omega t$$

$$T_{21} = 0$$

Vremenski prosjek za $\sin^2 \omega t$ je $\langle \sin^2 \omega t \rangle = \frac{1}{2}$.

Imamo,

$$\langle F_1 \rangle_{x=0} = -\frac{\epsilon_0 c^2}{4} \cdot \frac{\pi^2 E_0^2}{\omega^2 a^2} \underbrace{\int_0^a dy \sin^2\left(\frac{\pi}{a} y\right)}_{\frac{a}{2}} \underbrace{\int_0^h dz}_{h}$$

$$\langle F_1 \rangle_{x=0} = -\frac{\epsilon_0 c^2 \pi^2 E_0^2 h}{8\omega^2 a}$$

Uvrstimo li

$$\omega = \frac{c\pi}{a}\sqrt{2}$$

dobijemo

$$\langle F_1 \rangle_{x=0} = -\frac{\epsilon_0 a h E_0^2}{16}$$

Za $x=a$ odabir plohe je sličan, no sada je $n = -\vec{e}_x$

prema tome je na $x=a$

$$B_1(x=a) = 0$$

$$B_2(x=a) = -\frac{\pi E_0}{\omega a}\sin\left(\frac{\pi}{a}y\right)\sin\omega t$$

$$E_3(x=a) = 0$$

No, u rezultatu se javlja $B_2^2$ pa promjena predznaka nema utjecaj na konačni rezultat, samo ima predznak kao i ovdje.

Zato je

$$\langle F_1 \rangle_{x=a} = \frac{\epsilon_0 a h E_0^2}{16} = -\langle F_1 \rangle_{x=0}$$

Po $y=0$ je $\vec{n} = \vec{e}_y$; odnosno, $n_1 = n_3 = 0$ i $n_2 = 1$ pa je

$$\sum_j T_{ij} n_j = T_{i2} n_2 = T_{i2} = \epsilon_0 \left[ E_i E_2 + c^2 B_i \cdot B_2 - \frac{1}{2}\left(\vec{E}\cdot\vec{E} + c^2 \vec{B}\cdot\vec{B}\right)\delta_{i2} \right]$$

Računamo matrične elemente; $E_1 = E_2 = 0$; $B_3 = 0$

$$T_{12} = \epsilon_0 c^2 B_1 B_2$$

$$T_{22} = \epsilon_0 \left[ c^2 B_2^2 - \frac{1}{2}E_3^2 - \frac{1}{2}c^2(B_1^2 + B_2^2) \right]$$

$$= \epsilon_0 \left[ -\frac{1}{2}E_3^2 + \frac{1}{2}c^2 B_2^2 - \frac{1}{2}c^2 B_1^2 \right]$$

$T_{32} = 0$

Za $y=0$ imamo

$$B_1(y=0) = -\frac{\pi E_0}{\omega a} \sin\left(\frac{\pi}{a}x\right)\sin(\omega t)$$

$$B_2(y=0) = 0$$

$$E_3(y=0) = 0$$

Imamo

$$T_{22} = \epsilon_0 \cdot \left(-\frac{1}{2}\right) c^2 \left(\frac{\pi E_0}{\omega a}\right)^2 \sin^2\left(\frac{\pi}{a}x\right) \sin^2(\omega t)$$

$$\langle \sin^2 \omega t \rangle = \frac{1}{2}$$

$$\langle F_2 \rangle_{y=0} = -\frac{\epsilon_0 c^2}{4} \cdot \frac{\pi^2 E_0^2}{\omega^2 a^2} \int_0^a dx \sin^2\left(\frac{\pi}{a}x\right) \int_0^h dz$$

$$= -\frac{\epsilon_0 a h E_0^2}{16}$$

gdje smo uvrstili $\omega = \frac{c\pi}{a}\sqrt{2}$

Ostaje još stranica $z=0$. Na toj je stranici $\vec{n} = \vec{e}_z$, odnosno, $n_1 = n_2 = 0$ i $n_3 = 1$.

$$\sum_j T_{ij} n_j = T_{i3} n_3 = T_{i3} = \epsilon_0 \Big[ E_i E_3 + c^2 B_i B_3 - \frac{1}{2}\left(\vec{E}\cdot\vec{E} + c^2 \vec{B}\cdot\vec{B}\right)\delta_{i3} \Big]$$

Matrični elementi su:

$T_{13} = 0$

$T_{23} = 0$

$$T_{33} = \epsilon_0 \left[ E_3^2 - \frac{1}{2}\left( E_3^2 + c^2 (B_1^2 + B_2^2) \right) \right]$$

$$= \epsilon_0 \left[ \frac{1}{2} E_3^2 - \frac{1}{2} c^2 ( B_1^2 + B_2^2 ) \right]$$

Polja ne ovise o $z$. Uz $\langle \sin^2 \omega t \rangle = \langle \cos^2 \omega t \rangle = \frac{1}{2}$, za silu dobijemo

$$\langle F_3 \rangle_{z=0} = \frac{\epsilon_0}{2} \cdot \frac{1}{2} \cdot \Bigg[ E_0^2 \int_0^a dx \sin^2\left(\frac{\pi x}{a}\right) \int_0^a dy \sin^2\left(\frac{\pi y}{a}\right)$$

$$- c^2 \cdot \frac{\pi^2 E_0^2}{\omega^2 a^2} \int_0^a dx \sin^2\left(\frac{\pi x}{a}\right) \int_0^a dy \cos^2\left(\frac{\pi y}{a}\right)$$

$$- c^2 \cdot \frac{\pi^2 E_0^2}{\omega^2 a^2} \int_0^a dx \cos^2\left(\frac{\pi x}{a}\right) \int_0^a dy \sin^2\left(\frac{\pi y}{a}\right) \Bigg]$$

$$= \frac{\epsilon_0 E_0^2}{4} \left[ \frac{a^2}{4} - c^2 \frac{\pi^2}{\omega^2 a^2} \cdot \frac{a^2}{4} - c^2 \frac{\pi^2}{\omega^2 a^2} \cdot \frac{a^2}{4} \right]$$

Uvrstimo

$$\omega^2 = \frac{2 c^2 \pi^2}{a^2}$$

$$= 0$$

$\boxed{2.}$

Ukupna energija EM polja glasi ( Jackson, 6.106)

$$E(t) = \frac{\epsilon_0}{2}\int d^3r\left(\vec{E}\cdot\vec{E} + \frac{\vec{B}\cdot\vec{B}}{\epsilon_0\mu_0}\right) = \frac{\epsilon_0}{2}\int d^3r\left(\vec{E}\cdot\vec{E} + c^2\vec{B}\cdot\vec{B}\right)$$

$$\frac{1}{\epsilon_0\mu_0} = c^2 \; ; \quad \vec{E}, \vec{B} \text{ ovise o } \vec{r} \text{ i } t$$

Promotrit ću samo dio s električnim poljem. Polje $\vec{E}(\vec{r},t)$ možemo zapisati kao razvoj po ravnim valovima

$$\vec{E}(\vec{r},t) = \frac{1}{(2\pi)^3}\sum_{\lambda}\int d^3k\, \vec{E}_0(\vec{k})\, e^{i\vec{k}\cdot\vec{r}} \cdot e^{-i\omega t} \qquad (*)$$

gdje je $\vec{E}_0(\vec{k}) = E_0(\vec{k})\,\vec{e}_\lambda(\vec{k})$, $\omega = |\vec{k}|c$. Suma po $\lambda$ je suma po dvije polarizacije koje su okomite na vektor $\vec{k} = k\,\vec{e}_k$

$$\vec{e}_k \cdot \vec{e}_\lambda = 0 \quad , \; \lambda = 1,2$$

jer se radi o ravnom valu. Budući je $\vec{E}(\vec{r},t)$ realna funkcija mora vrijediti:

$$E_0^*(\vec{k})\, e^{i\omega t} = E_0(-k)\, e^{-i\omega t}$$

$$\int d^3r\; \vec{E}\cdot\vec{E} = \frac{1}{(2\pi)^6}\sum_{\lambda\lambda'}\int d^3k \int d^3k' \int d^3r\; E_0(\vec{k})\,E_0(\vec{k}')$$
$$\vec{e}_\lambda(\vec{k})\cdot\vec{e}_{\lambda'}(\vec{k}')\, e^{i\vec{k}\cdot\vec{r}} \cdot e^{i\vec{k}'\cdot\vec{r}} \cdot e^{-i\omega t} \cdot e^{-i\omega' t}$$

gdje je

$$\omega' = k'c \qquad \vec{e}_\lambda(-\vec{k}') = \vec{e}_\lambda(\vec{k}')$$

Promijenimo : $\vec{k}' \to -\vec{k}'$; $E_0(-\vec{k}') = E_0^*(\vec{k}')$; Jacobian pri promjeni koordinata je $-1$, ali kad daje 1. Imamo:

$$\frac{1}{(2\pi)^3}\int e^{i(\vec{k}-\vec{k}')\cdot\vec{r}}\, d^3r = \delta(\vec{k}-\vec{k}')$$

Delta funkcija dokida jedan integral po $\vec{k}'$. Također,

$$\vec{e}_{\lambda}(\vec{k}')\cdot\vec{e}_{\lambda}(\vec{k}) = \vec{e}_{\lambda'}(\vec{k})\cdot\vec{e}_{\lambda}(\vec{k}) = \delta_{\lambda\lambda'}$$

$$\int d^3r\, \vec{E}\cdot\vec{E} = \frac{1}{(2\pi)^3}\underbrace{\sum_{\lambda}}_{2}\int d^3k\, E_0(\vec{k})E_0^*(\vec{k})\,\underbrace{e^{-i\omega t}\cdot e^{+i\omega t}}_{=1}$$

$$= \frac{2}{(2\pi)^3}\int d^3k\, E_0(\vec{k})E_0^*(\vec{k})$$

Slično bismo dobili i za magnetsko polje s tim da treba paziti na vektor polarizacije: naime, mijenja predznak s transformacijom $\vec{k}'\to-\vec{k}'$

$$\vec{e}_{k'}\times\vec{e}_{\lambda} \longrightarrow -\vec{e}_{k'}\times\vec{e}_{\lambda}$$

$$\int d^3r\, \vec{B}\cdot\vec{B} = \frac{2}{(2\pi)^3}\int d^3k\, c^2 B_0(\vec{k})B_0^*(\vec{k})$$

<u>Broj fotona</u> dobijemo dijeljenjem gustoće

$$\frac{\epsilon_0}{2}\cdot\frac{2}{(2\pi)^3}\left(E_0(\vec{k})E_0^*(\vec{k}) + c^2 B_0(\vec{k})B_0^*(\vec{k})\right)$$

sa $\underbrace{\hbar ck}_{\hbar\omega(k)}$, te integriranjem

$$N = \frac{\epsilon_0}{2}\cdot\frac{2}{(2\pi)^3}\int d^3k\left(\frac{E_0(\vec{k})E_0^*(\vec{k})}{\hbar ck} + \frac{c^2 B_0(\vec{k})B_0^*(\vec{k})}{\hbar ck}\right)$$

Napišimo $E_0(\vec{k})\,E_0^*(\vec{k})$ u obliku

$$E_0(\vec{k})\,E_0^*(\vec{k}) = \left(\sum_\lambda E_0(\vec{k})\,\vec{e}_\lambda(\vec{k})\,e^{-i\omega t}\right)\cdot$$

$$\left(\sum_{\lambda'} E_0^*(\vec{k})\,\vec{e}_{\lambda'}(\vec{k})\,e^{i\omega t}\right)\cdot\frac{1}{2}$$

Faktor $\frac{1}{2}$ dolazi zbog toga što je

$$\vec{e}_\lambda(\vec{k})\cdot\vec{e}_{\lambda'}(\vec{k}) = \delta_{\lambda\lambda'}$$

pa ostaje sumiranje po $\lambda$.

S druge strane, iz (*) slijedi da je

$$\sum_\lambda E_0(\vec{k}')\,\vec{e}_\lambda(\vec{k}')\,e^{-i\omega' t} = \int d^3r\,\vec{E}(\vec{r},t)\,e^{-i\vec{k}'\cdot\vec{r}}$$

jer je $\int d^3r\, e^{i(\vec{k}-\vec{k}')\cdot\vec{r}} = (2\pi)^3\,\delta(\vec{k}-\vec{k}')$

Možemo zamijeniti

$$E_0(\vec{k})\,E_0^*(\vec{k}) = \frac{1}{2}\int d^3r\int d^3r'\,e^{-i\vec{k}\cdot(\vec{r}-\vec{r}')}\,\vec{E}(\vec{r},t)\cdot\vec{E}(\vec{r}',t)$$

Slična zamjena vrijedi i za $B_0(\vec{k})\,B_0^*(\vec{k})$

$$B_0(\vec{k})\,B_0^*(\vec{k}) = \frac{1}{2}\int d^3r\int d^3r'\,e^{-i\vec{k}\cdot(\vec{r}-\vec{r}')}\,\vec{B}(\vec{r},t)\cdot\vec{B}(\vec{r}',t)$$

Na kraju treba primijetiti da je integral po $\vec{k}$

$$\int \frac{e^{-i\vec{k}\cdot(\vec{r}-\vec{r}')}}{k}\,d^3k = \frac{4\pi}{|\vec{r}-\vec{r}'|^2}$$

Uvrstimo u konačan izraz

$$N = \underbrace{\frac{\epsilon_0}{8\pi^3}\cdot\frac{1}{2}\cdot\frac{4\pi}{\hbar c}}_{\frac{\epsilon_0}{4\pi^2\hbar c}}\int d^3r\int d^3r'\,\frac{\vec{E}\cdot\vec{E} + c^2\,\vec{B}\cdot\vec{B}}{|\vec{r}-\vec{r}'|^2}$$