# NAPREDNA ELEKTRODINAMIKA

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ZADATAK 1 Razmotrite točkasti magnetski moment $\mathbf{m}$ u gibajućem sustavu $K'$, s potencijalima $\Phi' = 0$ i $\mathbf{A}' = \mathbf{m} \times \mathbf{r}'/r'^3$ odnosno, u sustavu $K'$ postoji samo magnetsko polje.

(a) Pokažite da su potencijali do prvog reda po $\beta = v/c$ u sustavu $K$ jednaki

$$\Phi = \frac{(\boldsymbol{\beta}\times\mathbf{m})\cdot\mathbf{R}}{R^3}, \quad \mathbf{A} = \frac{\mathbf{m}\times\mathbf{R}}{R^3}$$

(b) Izračunajte električno i magnetsko polje u sustavu $K$ iz potencijala pod (a) te pokažite da se električno polje može napisati u obliku

$$\mathbf{E} = \mathbf{E}_{\text{dipole}}(\mathbf{p}_{\text{eff}} = \boldsymbol{\beta}\times\mathbf{m}) - \mathbf{m}\times\frac{[3\mathbf{n}(\mathbf{n}\cdot\boldsymbol{\beta}) - \boldsymbol{\beta}]}{R^3}$$

$$\mathbf{E} = \mathbf{E}_{\text{dipole}}\left(\mathbf{p}_{\text{eff}} = \frac{\boldsymbol{\beta}}{2}\times\mathbf{m}\right) + \frac{3}{2}\mathbf{n}\times\frac{[\mathbf{m}(\mathbf{n}\cdot\boldsymbol{\beta}) + \boldsymbol{\beta}(\mathbf{n}\cdot\mathbf{m})]}{R^3}$$

$$\mathbf{E} = \mathbf{B}\times\boldsymbol{\beta}$$

gdje je $\mathbf{B}$ magnetsko polje dipola.

ZADATAK 2 Pokažite da iz jednadžbe gibanja za elektromagnetsko polje

$$\frac{1}{4\pi}\partial^{\beta}F_{\beta\alpha} = \frac{1}{c}J_{\alpha}$$

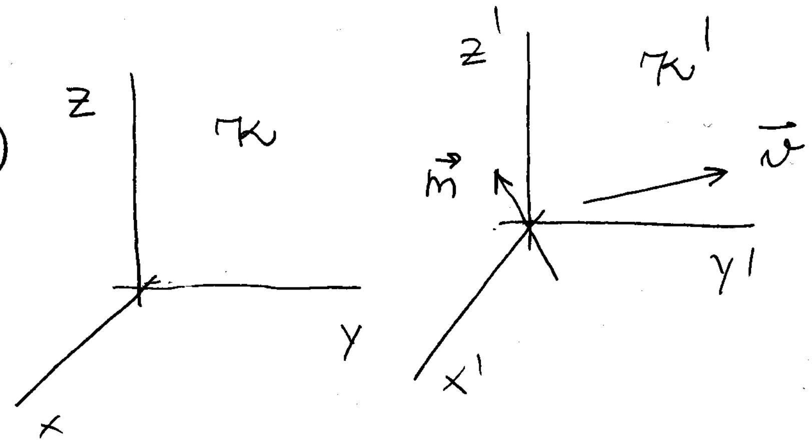
slijedi jednadžba kontinuiteta

$$\partial^{\alpha}J_{\alpha} = 0$$

***Uputa*:** pogledajte u Jacksonu, jedn. (12.90)

**1.**

(a)




$\vec m \simeq \vec m'$ do članova 1. reda po $\beta$ $\Rightarrow$ dokaz na kraju

4-vektor $A^\alpha = (A^0, \vec A)$ transformira se po formuli

$$\left.\begin{aligned} A_0' &= \gamma(A_0 - \vec\beta\cdot\vec A) \\ \vec A' &= \vec A + \frac{(\gamma-1)}{\beta^2}(\vec\beta\cdot\vec A)\vec\beta - \gamma\vec\beta A_0 \end{aligned}\right\} \quad (*)$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \simeq 1 + \frac{1}{2}\beta^2 \simeq 1$$ jer računamo do članova 1. reda po $\beta$

Inverzna transformacija za $(*)$; $\beta \to -\beta$

$$A_0 = \gamma(A_0' + \vec\beta\cdot\vec A')$$
$$\vec A = \vec A' + \frac{(\gamma-1)}{\beta^2}(\vec\beta\cdot A')\vec\beta + \gamma\vec\beta A_0'$$

U sustavu K' postoji samo magnetsko polje dipola magneta $\vec m$

$$\vec A' = m \times \frac{\vec r'}{r'^3}\,; \quad \Phi' = A_0' = 0$$

Imamo u sustavu K

$$A_0 = \vec\beta \cdot \frac{m \times \vec r'}{r'^3} = \frac{(\vec\beta \times \vec m)\vec r'}{r'^3}$$

$$\vec A = \vec m \times \frac{\vec r'}{r'^3}$$ (član $\frac{\gamma-1}{\beta^2}(\vec\beta\cdot\vec A')\vec\beta$ je 2. reda po $\beta$)

Još treba zamijeniti $\vec{r}\,'$. No,

$$\vec{r}\,' = \vec{r} + \frac{(\gamma-1)}{\beta^2}(\vec{\beta}\cdot\vec{r})\vec{\beta} - \gamma\vec{\beta}ct$$

do prvog reda po $\beta$ je to

$$\vec{r}\,' \simeq \vec{r} - \vec{\beta}ct = \underbrace{\vec{r} - \vec{v}t}_{\vec{R}}$$

Prema tome,

$$\Phi = A_0 = \frac{(\vec{\beta}\times\vec{m})\cdot\vec{R}}{R^3}$$

$$\vec{A} = \frac{\vec{m}\times\vec{R}}{R^3}$$

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(b)
$$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t}\cdot\frac{1}{c} \qquad (CGS)$$

$$\vec{\nabla}\Phi = \vec{\nabla}\left(\frac{(\vec{\beta}\times\vec{m})\cdot\vec{R}}{R^3}\right) = \left[(\vec{\beta}\times\vec{m})\cdot\vec{\nabla}\right]\left(\frac{\vec{R}}{R^3}\right) + (\vec{\beta}\times\vec{m})\times\left(\vec{\nabla}\times\frac{\vec{R}}{R^3}\right)$$

$$\vec{\nabla}\times\frac{\vec{R}}{R^3} = \vec{\nabla}\left(\frac{1}{R^3}\right)\times\vec{R} + \frac{1}{R^3}\underbrace{\vec{\nabla}\times\vec{R}}_{=\vec{\nabla}\times\vec{r}=0}$$

$$\left(\vec{\nabla}\left(\frac{1}{R^3}\right)\times\vec{R}\right)_i = \sum_{jk}\varepsilon_{ijk}\underbrace{\frac{\partial}{\partial x_j}\left(\frac{1}{R^3}\right)}_{-3R^{-4}\cdot\frac{1}{2}\cdot\frac{1}{R}\cdot 2(x_j - v_j t)}\cdot R_k$$

$$= -3\sum_{jk}\epsilon_{ijk}\frac{x_j - v_j t}{R^5}\cdot R_k$$

$$= -\frac{3}{R^5}(\vec{R}\times\vec{R})_i$$

$$\vec{\nabla}\frac{1}{R^3}\times\vec{R} = -\frac{3}{R^5}(\vec{R}\times\vec{R}) = 0$$

Ostaje član

$$\left[(\vec{z}\times\vec{m})\cdot\vec{\nabla}\right]\left(\frac{\vec{R}}{R^3}\right) = \sum_i \varepsilon_{ijk}\, z_j\, m_k \cdot \frac{\partial}{\partial x_i}\left(\frac{\vec{R}}{R^3}\right)$$

$$\frac{\partial}{\partial x_i}\left(\frac{\vec{R}}{R^3}\right) = \frac{1}{R^3}\frac{\partial}{\partial x_i}\vec{R} + \vec{R}\,\frac{\partial}{\partial x_i}\left(\frac{1}{R^3}\right)$$

$$= \frac{1}{R^3}\cdot\vec{e}_i + \vec{R}\cdot(-3)\cdot\frac{1}{R^4}\cdot\frac{1}{2}\cdot\frac{1}{R}\cdot 2(x_i - v_i t)$$

$$= \frac{\vec{e}_i}{R^3} - \frac{3\vec{R}}{R^5}\,R_i$$

Prema tome,

$$\sum_i \varepsilon_{ijk}\, z_j\, m_k\cdot\left(\frac{\vec{e}_i}{R^3} - \frac{3\vec{R}}{R^5}R_i\right) = \frac{(\vec{z}\times\vec{m})}{R^3} - \frac{3\left[(\vec{z}\times\vec{m})\cdot\vec{R}\right]\vec{R}}{R^5}$$

Označimo:

$$\vec{n} = \frac{\vec{R}}{R}$$

Član $-\vec{\nabla}\Phi$ postaje

$$\frac{3\left[(\vec{z}\times\vec{m})\cdot\vec{n}\right]\vec{n}}{R^3} - \frac{\vec{z}\times\vec{m}}{R^3} = \vec{E}_d \quad (\vec{p}_{el} = \vec{z}\times\vec{m})$$

Računamo član $-(1/c)\,\partial\vec{A}/\partial t$.

$$\frac{\partial\vec{A}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\vec{m}\times\vec{R}}{R^3}\right) = \frac{1}{R^3}\cdot\frac{\partial}{\partial t}(\vec{m}\times\vec{R}) + (\vec{m}\times\vec{R})\cdot\frac{\partial}{\partial t}\left(\frac{1}{R^3}\right)$$

$$\frac{\partial}{\partial t}(\vec{m}\times\vec{R}) = -\vec{m}\times\vec{v} = -c\,\vec{m}\times\vec{z}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{R^3}\right) = -3\cdot\frac{1}{R^4}\cdot\frac{1}{2}\cdot\frac{1}{R}\cdot 2\left[\sum_i (x_i - v_i t)\cdot(-v_i)\right]$$

$$\frac{\partial}{\partial t}\left(\frac{1}{R^3}\right) = \frac{3c}{R^5}\,\vec{R}\cdot\vec{\beta}$$

Prema tome, član $(-1/c)\,\partial\vec{A}/\partial t$ postaje

$$\frac{\vec{m}\times\vec{\beta}}{R^3} - (\vec{m}\times\vec{R})\cdot\frac{3}{R^5}(\vec{R}\cdot\vec{\beta})$$

$$= -\vec{m}\times\frac{[3\vec{n}(\vec{n}\cdot\vec{\beta})-\vec{\beta}]}{R^3}$$

Električno polje je konačno,

$$\vec{E} = \vec{E}_d\,(\vec{P}_{eff} = \vec{\beta}\times\vec{m}) - \vec{m}\times\frac{[3\vec{n}(\vec{n}\cdot\vec{\beta})-\vec{\beta}]}{R^3} \quad ; \quad \vec{R}\neq 0$$

Magnetsko polje

$$\vec{B} = \vec{\nabla}\times\vec{A} = \vec{\nabla}\times\left(\frac{\vec{m}\times\vec{R}}{R^3}\right)$$

$$= \vec{m}\left[\vec{\nabla}\left(\frac{\vec{R}}{R^3}\right)\right] - (\vec{m}\cdot\vec{\nabla})\left(\frac{\vec{R}}{R^3}\right)$$

Drugi član smo već izračunali

$$-(\vec{m}\cdot\vec{\nabla})\left(\frac{\vec{R}}{R^3}\right) = -\frac{\vec{m}}{R^3} + \frac{3(\vec{m}\cdot\vec{R})\vec{R}}{R^5} = \frac{3(\vec{m}\cdot\vec{n})\vec{n}-\vec{m}}{R^3}$$

Ostaje član

$$\vec{m}\left[\vec{\nabla}\left(\frac{\vec{R}}{R^3}\right)\right] = \vec{m}\cdot\left[\vec{R}\cdot\vec{\nabla}\left(\frac{1}{R^3}\right) + \frac{1}{R^3}\underbrace{\vec{\nabla}\cdot\vec{R}}_{3}\right]$$

$$\vec{\nabla}\left(\frac{1}{R^3}\right) = -3\,\frac{1}{R^4}\cdot\frac{1}{2}\cdot\frac{2\vec{R}}{R} = -3\,\frac{\vec{R}}{R^5}$$

Ostaje,

$$\vec{m}\left[\vec{\nabla}\left(\frac{\vec{R}}{R^3}\right)\right] = +\vec{m}\left[-3\,\frac{\vec{R}^2}{R^5} + 3\cdot\frac{1}{R^3}\right] = 0$$

↳ NAPOMENA

Na kraju je magnetsko polje, $\vec{R} \neq 0$

$$\vec{B} = \frac{3(\vec{m}\cdot\vec{n})\vec{n} - \vec{m}}{R^3} \quad //$$

To je polje magnetskog dipola!

NAPOMENA: Treba imati na umu da je ustvari

$$\vec{\nabla}\left(\frac{\vec{R}}{R^3}\right) = \vec{\nabla}\cdot\vec{\nabla}\left(\frac{1}{R}\right)\cdot(-1) = -\vec{\nabla}^2\frac{1}{R} = 4\pi\delta(\vec{R})$$

Ali, recimo da računamo u točkama $\vec{R} \neq 0$

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Treba pokazati da je $\vec{E} = \vec{B}\times\vec{z}$.

$$\vec{B}\times\vec{z} = \frac{1}{R^3}\left[3(\vec{m}\cdot\vec{n})(\vec{n}\times\vec{z}) - \vec{m}\times\vec{z}\right]$$

Raspišimo najprije izraz

$$\vec{m}\times\left[\vec{n}\times(\vec{n}\times\vec{z})\right]$$

na dva načina.

$$1. \quad \vec{m}\times\left[\vec{n}\times(\vec{n}\times\vec{z})\right] = \vec{n}\left[\vec{m}\cdot(\vec{n}\times\vec{z})\right] - (\vec{m}\cdot\vec{n})(\vec{n}\times\vec{z})$$

$$\parallel$$

$$2. \quad \vec{m}\times\left[\vec{n}\times(\vec{n}\times\vec{z})\right] = \vec{m}\times\left[\vec{n}(\vec{z}\cdot\vec{n}) - \vec{z}\underbrace{(\vec{n}\cdot\vec{n})}_{=1}\right]$$

$$= (\vec{m}\times\vec{n})(\vec{z}\cdot\vec{n}) - \vec{m}\times\vec{z}$$

U izrazu za $\vec{B}\times\vec{z}$ pojavljuje se $3(\vec{m}\cdot\vec{n})(\vec{n}\times\vec{z})$

$$3(\vec{m}\cdot\vec{n})(\vec{n}\times\vec{z}) = 3\vec{n}\left[\vec{m}\cdot(\vec{n}\times\vec{z})\right] - 3(\vec{m}\times\vec{n})(\vec{z}\cdot\vec{n}) + 3\,\vec{m}\times\vec{z} \qquad (*)$$

Zbog $\vec{m}\cdot(\vec{n}\times\vec{z}) = -\vec{m}\cdot(\vec{z}\times\vec{n}) = -(\vec{m}\times\vec{z})\cdot\vec{n} = (\vec{z}\times\vec{m})\cdot\vec{n}$

Imamo,

$$\vec{B}\times\vec{z} = \frac{1}{R^3}\left\{3\vec{n}\left[(\vec{z}\times\vec{m})\cdot\vec{n}\right] - 3(\vec{m}\times\vec{n})(\vec{n}\cdot\vec{z}) + \overbrace{3\,\vec{m}\times\vec{z}}^{-3\,\vec{z}\times\vec{m}} - \underbrace{\vec{m}\times\vec{z}}_{-\vec{z}\times\vec{m}}\right\}$$

$$= \underbrace{\frac{1}{R^3}\left\{3\vec{n}\left[(\vec{z}\times\vec{m})\cdot\vec{n}\right] - \vec{z}\times\vec{m}\right\}}_{\vec{E}_d(\vec{P}_{eff} = \vec{z}\times\vec{m})} +$$

$$+ \frac{1}{R^3}\left\{-3(\vec{m}\times\vec{n})(\vec{n}\cdot\vec{z}) - \vec{z}\times\vec{m}\right\}$$

$$= \vec{E}$$

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Još treba pokazati da vrijedi izraz

$$\vec{E} = \vec{E}_d\left(\vec{P}_{eff} = \frac{\vec{z}}{2}\times\vec{m}\right) + \frac{3}{2}\vec{n}\times\frac{\left[\vec{m}(\vec{n}\cdot\vec{z}) + \vec{z}(\vec{n}\cdot\vec{m})\right]}{R^3}$$

$$\vec{E}_d(\vec{P}_{eff} = \vec{z}\times\vec{m}) = 2\cdot\left\{\frac{3\left[(\vec{z}/2\times\vec{m})\cdot\vec{n}\right]\vec{n}}{R^3} - \frac{\vec{z}/2\times\vec{m}}{R^3}\right\}$$

$$= 2\cdot\vec{E}_d\left(\vec{P}_{eff} = \frac{\vec{z}}{2}\times\vec{m}\right)$$

Sada treba srediti izraz

$$\vec{E}_d\left(\vec{P}_{eff} = \frac{\vec{z}}{2}\times\vec{m}\right) - \vec{m}\times\frac{\left[3\vec{n}(\vec{n}\cdot\vec{z}) - \vec{z}\right]}{R^3}$$

jer je

$$\vec{E} = 2\vec{E}_d\left(\vec{P}_{eff} = \frac{\vec{z}}{2}\times\vec{m}\right) - \vec{m}\times\frac{\left[3\vec{n}(\vec{n}\cdot\vec{z}) - \vec{z}\right]}{R^3}$$

Izraz kojeg treba srediti je

$$\frac{3}{2}\cdot\frac{1}{R^3}\left[(\vec{z}\times\vec{m})\cdot\vec{n}\right]\vec{n}-\frac{3}{2}\cdot\frac{1}{R^3}\,\vec{z}\times\vec{m}-3\cdot\frac{1}{R^3}(\vec{m}\times\vec{n})(\vec{n}\cdot\vec{z}) \qquad (**)$$

Pogledajmo samo prva dva člana u gornjem izrazu. Izlučimo $\frac{3}{2}\cdot\frac{1}{R^3}$ te koristimo $(*)$ što smo prije dokazali

$$\left[\underbrace{(\vec{z}\times\vec{m})}\cdot\vec{n}\right]\vec{n}-\vec{z}\times\vec{m}=\left[\vec{m}\cdot(\vec{n}\times\vec{z})\right]n-\vec{z}\times\vec{m}$$

$$-\vec{n}\left[\vec{m}\cdot(\vec{z}\times\vec{n})\right]=-\vec{n}\left[(\vec{m}\times\vec{z})\cdot\vec{n}\right]$$

$$=(\vec{m}\cdot\vec{n})(\vec{n}\times\vec{z})+(\vec{z}\cdot\vec{n})(\vec{m}\times\vec{n})$$

Izraz $(**)$ postaje

$$\frac{3}{2}\cdot\frac{1}{R^3}\left\{(\vec{m}\cdot\vec{n})(\vec{n}\times\vec{z})-(\vec{z}\cdot\vec{n})(\vec{n}\times\vec{m})\right\}+\frac{3}{R^3}(\vec{n}\times\vec{m})(\vec{z}\cdot\vec{n})$$

gdje smo upotrijebili $\vec{n}\times\vec{m}=-\vec{m}\times\vec{n}$. Imamo konačno

$$\frac{3}{2}\cdot\frac{1}{R^3}\left\{(\vec{n}\times\vec{m})(\vec{z}\cdot\vec{n})+(\vec{n}\times\vec{z})(\vec{m}\cdot\vec{n})\right\}$$

čime smo pokazali početni izraz.

$\boxed{2.}$

Na jednadžbu

$$\frac{1}{4\pi}\partial^{\beta}F_{\beta\alpha} = \frac{1}{c}J_{\alpha}$$

djelovat ćemo s $\partial^{\alpha}$ (4-divergencija)

$$\frac{1}{4\pi}\partial^{\alpha}\partial^{\beta}F_{\beta\alpha} = \frac{1}{c}\partial^{\alpha}J_{\alpha}$$

Na lijevoj strani je

$$\frac{1}{4\pi}\left(\frac{1}{2}\partial^{\alpha}\partial^{\beta}F_{\beta\alpha} + \frac{1}{2}\underbrace{\partial^{\alpha}\partial^{\beta}}_{\alpha\leftrightarrow\beta}F_{\beta\alpha}\right)$$

$$= \frac{1}{4\pi}\left(\frac{1}{2}\partial^{\alpha}\partial^{\beta}\underbrace{F_{\beta\alpha}}_{-F_{\alpha\beta}} + \frac{1}{2}\partial^{\beta}\partial^{\alpha}F_{\alpha\beta}\right) =$$

$$= \frac{1}{4\pi}\left(-\frac{1}{2}\partial^{\beta}\partial^{\alpha}F_{\alpha\beta} + \frac{1}{2}\partial^{\beta}\partial^{\alpha}F_{\alpha\beta}\right) = 0$$

Prema tome,

$$\partial^{\alpha}J_{\alpha} = 0$$