# NAPREDNA ELEKTRODINAMIKA

## Treći kolokvij 18. 1. 2024.

ZADATAK 1 (a) Pomoću Fourierove superpozicije različitih frekvencija ili ekvivalentnom metodom, pokažite da je za realni električni dipolni moment $\mathbf{p}(t)$ trenutna snaga zračenja po jediničnom prostornom kutu na udaljenosti $r$ od dipola u smjeru $\mathbf{n} = \mathbf{e}_r$ jednaka

$$\frac{dP(t)}{d\Omega} = \frac{Z_0}{16\pi^2 c^2}\left|\left[\mathbf{n}\times\frac{d^2\mathbf{p}(t')}{dt'^2}\right]\times\mathbf{n}\right|^2$$

gdje je $t' = t - r/c$ retardirano vrijeme. Ponovite ukratko račun za magnetski moment $\mathbf{m}(t)$ i pokažite da se također dobije gornja formula uz zamjenu

$$(\mathbf{n}\times\ddot{\mathbf{p}})\times\mathbf{n} \to \frac{1}{c}\ddot{\mathbf{m}}\times\mathbf{n}$$

(b) Kao i pod (a), izračunajte trenutnu snagu zračenja po jediničnom prostornom kutu za realni tenzor kvadrupolnog momenta $Q_{\alpha\beta}(t)$

$$Q_{\alpha\beta}(t) = \int\left(3x_\alpha x_\beta - r^2\delta_{\alpha\beta}\right)\rho(\mathbf{r},t)\,d^3r$$

za realnu gustoću naboja $\rho(\mathbf{r}, t)$. Rezultat koji trebate dobiti je:

$$\frac{dP(t)}{d\Omega} = \frac{Z_0}{576\pi^2 c^4}\left|\left[\mathbf{n}\times\frac{d^3\mathbf{Q}(\mathbf{n},t')}{dt'^3}\right]\times\mathbf{n}\right|^2$$

Komponente vektora $\mathbf{Q}(\mathbf{n}, t)$ dane su formulom

$$Q_\alpha = \sum_\beta Q_{\alpha\beta} n_\beta$$

***Uputa*:** primijetite da se Fourierov transformat vektorskog potencijala po vremenu može zapisati u obliku

$$\mathbf{A}(\mathbf{r},\omega) = \frac{i\mu_0\omega}{4\pi}\frac{e^{i(\omega/c)r}}{r}\mathbf{K}(\omega)$$

gdje je $\mathbf{K}(\omega) \to -\mathbf{p}(\omega)$ za električni dipol, $\mathbf{K}(\omega) \to (1/c)\mathbf{n}\times\mathbf{m}(\omega)$ za magnetski dipol te $\mathbf{K}(\omega) \to i(\omega/6c)\mathbf{Q}(\omega)$ električni kvadrupol (pogl. Jackson, str. 410-414). Trenutna snaga zračenja po jediničnom prostornom kutu računa se pomoću formule:

$$\frac{dP(t)}{d\Omega} = r^2\mathbf{n}\cdot\left[\mathbf{E}(t)\times\mathbf{H}(t)\right]$$

ZADATAK 2 Nađite diferencijalni udarni presjek u Bornovoj aproksimaciji za raspršenje EM zračenja na česticama Drudeove plazme čija gustoća eksponencijalno pada s udaljenosti od ishodišta

$$n(r) = n_0 e^{-\kappa r}$$

gdje su $n_0 > 0$ i $\kappa > 0$ konstante.

$\boxed{1.}$

Trenutna snaga zračenja po jediničnom prostornom kutu

$$\frac{dP(t)}{d\Omega} = r^2 \vec{e}_r \cdot \left[\vec{E}(t) \times \vec{H}(t)\right] \quad ; \quad \boxed{\vec{e}_r = \vec{n}}$$

Polja $\vec{E}(t)$ i $\vec{H}(t)$ su realna. Uzmemo li inverzni Fourierov transformat

$$\vec{E}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \vec{E}(\omega) e^{-i\omega t} d\omega$$

$$\vec{H}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \vec{H}(\omega) e^{-i\omega t} d\omega$$

gdje zbog realnosti polja $\vec{E}$ i $\vec{H}$ mora vrijediti

$$\vec{E}(-\omega) = \vec{E}^*(\omega)$$

$$\vec{H}(-\omega) = \vec{H}^*(\omega)$$

Za električni dipol, Fourierov transformat vektorskog potencijala glasi

$$\vec{A}(\vec{r},\omega) = i \frac{\mu_0 \omega}{4\pi} \cdot \frac{e^{i(\omega/c)r}}{r} \cdot (-\vec{p})$$

(Jackson, 9.16) Iz ove formule onda možemo naći Fourierove komponente za $\vec{E}$ i $\vec{H}$

$$\vec{H}(\vec{r},\omega) = \frac{1}{\mu_0} \vec{\nabla} \times \vec{A}(\vec{r},\omega)$$

$$\vec{\nabla} \times \vec{A} = \vec{\nabla} \times \left( \frac{e^{i(\omega/c)r}}{r} \vec{p} \right) = \vec{\nabla}\left(\frac{e^{i(\omega/c)r}}{r}\right) \times \vec{p} + \frac{e^{i(\omega/c)r}}{r} \underbrace{\vec{\nabla} \times \vec{p}}_{=0}$$

$$\vec{\nabla}\left(\frac{e^{i(\omega/c)r}}{r}\right) = \frac{\partial}{\partial r}\left(\frac{e^{i(\omega/c)r}}{r}\right)\vec{e_r}$$

$$= \left[e^{i(\omega/c)r} \cdot i(\omega/c) \cdot \frac{1}{r} - e^{i(\omega/c)} \cdot \frac{1}{r^2}\right]\vec{e_r}$$

Zadržat ćemo samo 1. član, taj ide u zračenje.

Prema tome,

$$\vec{H} = \frac{-1}{\mu_0} \cdot i\,\frac{\mu_0\omega}{4\pi} \cdot i\,\frac{\omega}{c} \cdot \frac{1}{r}\, e^{i(\omega/c)r}\, \vec{e_r} \times \vec{P}$$

$$= \frac{\omega^2}{4\pi c}\left(\vec{n} \times \vec{P_\omega}\right)\frac{e^{i(\omega/c)r}}{r}$$

Električno polje (u zoni zračenja)

$$\vec{E} = \sqrt{\frac{\mu_0}{\epsilon_0}} \cdot \frac{\omega^2}{4\pi c} \cdot \frac{e^{i(\omega/c)r}}{r} \cdot \left(\vec{n} \times \vec{P_\omega}\right) \times \vec{n}$$

$$c = \frac{1}{\sqrt{\mu_0\epsilon_0}}\ ; \quad \vec{E} = \frac{\mu_0\omega^2}{4\pi} \cdot \frac{e^{i(\omega/c)r}}{r}\left(\vec{n} \times \vec{P_\omega}\right) \times \vec{n}$$

Sada možemo računati raspodjelu snage po prostornom kutu

$$\frac{dP}{d\Omega} = \frac{r^2}{(2\pi)^2}\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega'\, e^{-i(\omega+\omega')t} \cdot \left(\frac{\mu_0\omega^2}{4\pi} \cdot \frac{\omega'^2}{4\pi c}\right)$$

Fourier transform

$$e^{i(\omega+\omega')r/c} \cdot \frac{1}{r^2}\ \vec{n} \cdot \left[\left(\left(\vec{n} \times \vec{P_\omega}\right) \times \vec{n}\right) \times \left(\vec{n} \times \vec{P_{\omega'}}\right)\right]$$

Sada treba izmnožiti vektore u zagradi. Vektor $\vec{P_\omega}$ je ustvari $\vec{P}(\omega)$.

$$\vec{n}\cdot\left[(\vec{n}\times\vec{p}_\omega)\times\vec{n}\ \times(\vec{n}\times\vec{p}_{\omega'})\right]$$

$$= \vec{n}\cdot\left\{(\vec{n}\times\vec{p}_\omega)\cdot(\vec{n}\times\vec{p}_{\omega'})\,\vec{n} - [\vec{n}\cdot\underbrace{(\vec{n}\times\vec{p}_{\omega'})}_{=0}](\vec{n}\times\vec{p}_\omega)\right\}$$

$$= \underbrace{(\vec{n}\cdot\vec{n})}_{=1}\left[(\vec{n}\times\vec{p}_\omega)\cdot(\vec{n}\times\vec{p}_{\omega'})\right]$$

$$= \underbrace{(\vec{n}\cdot\vec{n})}_{=1}(\vec{p}_\omega\cdot\vec{p}_{\omega'}) - (\vec{n}\cdot\vec{p}_\omega)(\vec{n}\cdot\vec{p}_{\omega'})$$

Integral postaje

$$\frac{dP}{d\Omega} = \frac{\mu_0}{16c\pi^2}\int_{-\infty}^{\infty}d\omega\int_{-\infty}^{\infty}d\omega'\, e^{-i(\omega+\omega')(t-\frac{r}{c})}\,\omega^2\omega'^2$$

$$\cdot\left[\vec{p}_\omega\cdot\vec{p}_{\omega'} - (\vec{n}\cdot\vec{p}_\omega)(\vec{n}\cdot\vec{p}_{\omega'})\right]$$

Fourierov transformat za $\vec{p}(t)$

$$\vec{p}(\omega) = \int_{-\infty}^{\infty} dt\, \vec{p}(t)\, e^{i\omega t}$$

Inverzni Fourierov transformat

$$\vec{p}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \vec{p}(\omega)\, e^{-i\omega t}$$

$$\frac{d^2\vec{p}}{dt^2} = -\frac{\omega^2}{2\pi}\int_{-\infty}^{\infty} d\omega\, \vec{p}(\omega)\, e^{-i\omega t} = -\omega^2\vec{p}(t)$$

$$\vec{p}(t) = \frac{1}{-\omega^2}\ddot{\vec{p}} \Rightarrow$$ uvrstimo u Fourierov transformat

$$\vec{p}(\omega) = \frac{1}{-\omega^2} \int_{-\infty}^{\infty} dt\, \ddot{\vec{p}}(t)\, e^{i\omega t}$$

Ispod integrala se upravo javlja $\omega^2 \vec{p}(\omega)$. Uzmimo prvi član i integraciju po $\omega$

$$-\int_{-\infty}^{\infty} d\omega\, e^{-i\omega(t-\frac{r}{c})} \int_{-\infty}^{\infty} dt'\, \ddot{\vec{p}}(t')\, e^{i\omega t'}$$

$$= -\int_{-\infty}^{\infty} dt'\, \ddot{\vec{p}}(t') \underbrace{\int_{-\infty}^{\infty} d\omega\, e^{i\omega[t'-(t-\frac{r}{c})]}}_{2\pi\,\delta(t'-(t-\frac{r}{c}))}$$

$$= -2\pi\, \ddot{\vec{p}}\left(t-\frac{r}{c}\right)$$

$\hookrightarrow$ ovo je ustvari $\ddot{\vec{p}}(t')\Big|_{t'=t-\frac{r}{c}}$

Jednak račun vrijedi i za $\omega'^2 \vec{p}(\omega')$, a slično je i za $\omega^2\, \vec{n}\cdot\vec{p}(\omega)$ i $\omega'^2\, \vec{n}\cdot\vec{p}(\omega')$. Prema tome,

$$\frac{dP}{d\Omega} = \frac{1}{\cancel{(2\pi)^2}} \cdot \cancel{(2\pi)^2} \cdot \frac{\mu_0}{16 c \pi^2} \cdot \left[ \ddot{\vec{p}}(t')^2 - (\vec{n}\cdot\ddot{\vec{p}}(t'))^2 \right]$$

gdje je $t' = t - \frac{r}{c}$. Zbog

$$\ddot{\vec{p}}(t')^2 - (\vec{n}\cdot\ddot{\vec{p}}(t'))^2 = [\underbrace{(\vec{n}\times\ddot{\vec{p}}(t')]\times\vec{n}}_{[\ddot{\vec{p}}-(\vec{n}\cdot\ddot{\vec{p}})\vec{n}]^2}]^2$$

dobivamo konačan rezultat

$$\frac{dP(t)}{d\Omega} = \frac{\mu_0}{16 c \pi^2} \left| \left[ \vec{n} \times \ddot{\vec{p}}(t') \right] \times \vec{n} \right|^2$$

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \; ; \quad \frac{\mu_0}{c} = \frac{Z_0}{c^2} \; ; \quad Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$$

Za magnetni dipol, treba krenuti od izraza

$$\vec{A}(\vec{r}, \omega) = \frac{i\omega\mu_0}{4\pi c} \cdot \frac{e^{i(\omega/c)r}}{r} (\vec{n} \times \vec{m})$$

(Jackson, 9.33). Pri tome je $\vec{m}$ ustvari $\vec{m}(\omega)$.

Uvrstimo li u konačni izraz

$$\ddot{\vec{p}}(t') \rightarrow -\frac{1}{c} (\vec{n} \times \ddot{\vec{m}}(t'))$$

imamo

$$[\vec{n} \times \ddot{\vec{p}}] \times \vec{n} \rightarrow -\frac{1}{c} \left[ \vec{n} \times (\vec{n} \times \ddot{\vec{m}}) \right] \times \vec{n}$$

$$= \frac{1}{c} \left[ (\vec{n} \cdot \ddot{\vec{m}}) \vec{n} - \ddot{\vec{m}} \right] \times \vec{n}$$

$$= -\frac{1}{c} \left[ (\vec{n} \cdot \ddot{\vec{m}}) \underbrace{\vec{n} \times \vec{n}}_{=0} - \ddot{\vec{m}} \times \vec{n} \right]$$

Dobijemo

$$\frac{dP(t)}{d\Omega} = \frac{Z_0}{16 \pi^2 c^2} \cdot \frac{1}{c^2} \left| \vec{n} \times \ddot{\vec{m}}(t') \right|^2$$

(b) Vektorski potencijal magnetnog kvadrupola koji opisuje zračenje je oblika

$$\vec{A}(\vec{r}, \omega) = -\frac{\mu_0 \omega^2}{24 \pi c} \cdot \frac{e^{i(\omega/c)r}}{r} \vec{Q}(\omega)$$

gdje je $\vec{Q}$ definiran pomoću

$$\vec{n} \times \int \vec{r}\,'(\vec{n}\cdot\vec{r}\,')\rho(\vec{r}\,')\,d^3r' = \frac{1}{3}\,\vec{n}\times\vec{Q}$$

i to je vektor čije su komponente

$$Q_i = \sum_j \tilde{Q}_{ij}\, n_j$$

$\tilde{Q}_{ij}$ je tenzor kvadrupolnog momenta

$$\tilde{Q}_{ij} = \int (3x_i x_j - r^2\delta_{ij})\,\rho(\vec{r})\,d^3r$$

Kada izračunamo polja $\vec{E}$ i $\vec{H}$, odatle će se pojaviti ispod integrala $\omega^3$. To upućuje da treba tražiti 3. derivaciju

$$\vec{Q}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \vec{Q}(\omega)\, e^{-i\omega t}$$

$$\frac{d^3Q}{dt^3} = +\frac{i\omega^3}{2\pi}\int_{-\infty}^{\infty} d\omega\, \vec{Q}(\omega)\, e^{-i\omega t} = +i\omega^3\,\vec{Q}(t)$$

i zamijeniti

$$\vec{Q}(\omega) = \frac{1}{+i\omega^3}\int_{-\infty}^{\infty} dt\, \dddot{\vec{Q}}(t)\, e^{i\omega t}$$

Dalje je postupak identičan; upotrebljavaju se Diracove delta funkcije. Prema tome, u izrazu za raspodjelu snage za električni dipol, treba napraviti zamjenu

$$\ddot{\vec{p}}(t') \rightarrow +\frac{i}{6c}\,\dddot{\vec{Q}}(t')$$

Imamo

$$[\vec{n} \times \ddot{\vec{p}}] \times \vec{n} \rightarrow +i\left(\frac{1}{6c}\right)[\vec{n} \times \dddot{Q}] \times \vec{n}$$

pa je snaga zračenja za kvadrupol

$$\frac{dP(t)}{d\Omega} = \frac{\mu_0}{16 c \pi^2} \underbrace{\left| +i \frac{1}{6c} \right|^2}_{\frac{1}{36c^2}} \left| [\vec{n} \times \dddot{Q}(t')] \times \vec{n} \right|^2$$

$$= \frac{\mu_0}{576 c^3 \pi^2} \left| [n \times \dddot{Q}(t')] \times \vec{n} \right|^2$$

$$\frac{\mu_0}{c} = \frac{Z_0}{c^2}$$ kao i prije!

$\boxed{2.}$

Diferencijalni udarni presjek za raspršenje na česticama Prudeve plazme u Bornovoj aproksimaciji, glasi:

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{4\pi\varepsilon_0 mc^2}\right)^2 |\vec{e}_k \times \vec{e}_0|^2 \left| \int d^3r' n(\vec{r}') \exp[i(\vec{k}_0 - \vec{k})\cdot\vec{r}'] \right|^2$$

Računamo integral; $\vec{q} \equiv \vec{k} - \vec{k}_0$; postavimo os $z$ u smjeru vektora $\vec{q}$. Imamo

$$n_0 \int d^3r' e^{-\varkappa r'} \cdot \exp[iqr'\cos\theta']$$

$$= n_0 \int_0^{2\pi} d\phi' \int_0^{\infty} dr' r'^2 e^{-\varkappa r'} \int_0^{\pi} d\theta \sin\theta' \exp[iqr'\cos\theta']$$

$$\int_0^{\pi} d\theta \sin\theta' \exp[iqr'\cos\theta'] = \int_{-1}^{1} d(\cos\theta) \exp[iqr'\cos\theta']$$

$$= \frac{\exp[iqr'\cos\theta']}{iqr'}\Bigg|_{-1}^{1} = \frac{1}{iqr'}\Big(\underbrace{e^{iqr'} - e^{-iqr'}}_{2i\sin(qr')}\Big)$$

Integral je jednak

$$n_0 \cdot 2\pi \cdot \frac{2}{q} \underbrace{\int_0^{\infty} dr' r' e^{-\varkappa r'} \cdot \sin(qr')}_{\operatorname{Im} \int_0^{\infty} dr' r' \exp[-\varkappa r' + iqr']}$$

Bronštejn!

$$= \operatorname{Im} \frac{\exp[-\varkappa r' + iqr']}{(-\varkappa + iq)^2}\Big[(-\varkappa + iq)r' - 1\Big]\Bigg|_0^{\infty}$$

$$= \operatorname{Im} \frac{1}{(-\varkappa + iq)^2}$$

2-1

$$\mathrm{Im}\ \frac{(-\varkappa - iq)^2}{(\varkappa^2+q^2)^2} = \frac{2\varkappa q}{(\varkappa^2+q^2)^2}$$

Integral je konačno,

$$8\pi i\ \frac{n_0 \varkappa}{(\varkappa^2+q^2)^2}$$

Za nepolarizirani upadni val je (prosjek!)

$$|\vec{e}_k \times \vec{e}_0|^2 = \frac{1}{2}(1+\cos^2\theta)$$

pa je

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{4\pi\epsilon_0 mc^2}\right)^2 \cdot \frac{1}{2}(1+\cos^2\theta)\cdot(8\pi n_0 \varkappa)^2 \frac{1}{(\varkappa^2+q^2)^4}$$

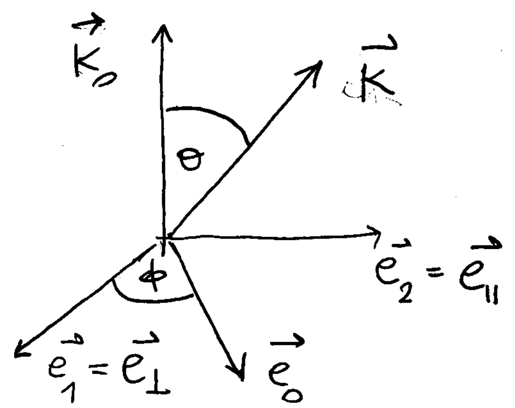
$$q = 2k_0 \sin(\theta/2)$$

$$\frac{d\sigma}{d\Omega} = 2\cdot\left(\frac{e^2 n_0}{\epsilon_0 mc^2 \varkappa^3}\right)^2 \cdot \frac{1+\cos^2\theta}{\left[1+4(k_0/\varkappa)^2 \sin^2\theta/2\right]^4}$$

gdje je $\vec{k}_0$ u smjeru $z$.

Ako je upadni val linearno polariziran

$$|\vec{e}_k \times \vec{e}_0|^2 = 1-(\vec{e}_k\cdot\vec{e}_0)^2 =$$




$$\vec{e}_0 = \cos\phi\, \vec{e}_1 + \sin\phi\, \vec{e}_2$$

$$\vec{e}_k = \frac{\vec{k}}{k}$$

$$\vec{e}_k\cdot\vec{e}_0 = \sin\phi\, \vec{e}_2\cdot\vec{e}_k = \sin\phi \sin\theta$$

$$\vec{e}_k\cdot\vec{e}_1 = 0$$

$$|\vec{e}_K \times \vec{e}_0|^2 = 1 - \sin^2\phi \sin^2\theta$$

pa je

$$\frac{d\sigma}{d\Omega} = \left(\frac{4e^2 n_0}{\epsilon_0 m c^2 \varkappa^3}\right)^2 \cdot \frac{1 - \sin^2\phi \sin^2\theta}{\left[1 + 4(K_0/\varkappa)^2 \sin^2(\theta/2)\right]^4}$$