SVEUČILIŠTE U RIJECI, FAKULTET ZA FIZIKU

Diplomski studij Fizika **Ak. god. 2023./2024.**

NAPREDNA KVANTNA MEHANIKA

Treći kolokvij 1. 2. 2024.

ZADATAK 1 Operatore stvaranja i poništenja ***za bozone*** definiramo na sljedeći način:

$$a_i^\dagger |n_1 n_2 ... n_i ... n_\infty\rangle = \sqrt{n_i + 1}\, |n_1 n_2 ..., n_i + 1, ... n_\infty\rangle$$

$$a_i |n_1 n_2 ... n_i ... n_\infty\rangle = \sqrt{n_i}\, |n_1 n_2 ..., n_i - 1, ... n_\infty\rangle$$

Pokažite da za operatore stvaranja i poništenja vrijede komutacijske relacije

$$\left[a_k, a_l^\dagger\right] = a_k a_l^\dagger - a_l^\dagger a_k = \delta_{kl}$$

$$\left[a_k, a_l\right] = \left[a_k^\dagger, a_l^\dagger\right] = 0$$

ZADATAK 2 (a) Na vježbama smo spomenuli da operator

$$\mathbf{\Sigma} \cdot \hat{\mathbf{n}}$$

gdje je $\hat{\mathbf{n}}$ proizvoljan jedinični vektor, ne komutira s Diracovim hamiltonijanom. Nasuprot tome, operator heliciteta

$$\mathbf{\Sigma} \cdot \hat{\mathbf{p}}$$

komutira s Diracovim hamiltonijanom, gdje je $\hat{\mathbf{p}}$ jedinični vektor u smjeru operatora impulsa. Dokažite ovu tvrdnju.

(b) Pokažite da su ± 1 svojstvene vrijednosti operatora heliciteta.

(c) Pokažite da je operator helicteta rotacijski invarijantan.

$\boxed{1.}$ Neka je $l > k$. Vektor baze u Fockovom prostoru $\{|n_1 n_2 \dots n_\infty\rangle\}$.

$$a_k a_l^+ |n_1 n_2 \dots n_\infty\rangle = a_k \left(\sqrt{n_l+1}\, |n_1, \dots, n_l+1, \dots, n_\infty\rangle\right.$$

$$= \sqrt{n_l+1} \cdot \sqrt{n_k}\, |n_1, \dots, n_k-1, \dots, n_l+1, \dots n_\infty\rangle$$

$$a_l^+ a_k |n_1 n_2 \dots n_\infty\rangle = a_l^+ \left(\sqrt{n_k}\, |n_1, \dots, n_k-1, \dots n_\infty\rangle\right.$$

$$= \sqrt{n_k}\sqrt{n_l+1}\, |n_1, \dots, n_k-1, \dots, n_l+1, \dots n_\infty\rangle$$

Budući je $l > k$, $l \neq k$ imamo

$$(a_k a_l^+ - a_l^+ a_k) |n_1 n_2 \dots n_\infty\rangle = 0$$

Neka je $l = k$. Tada je

$$a_k a_k^+ |n_1 \dots n_\infty\rangle = a_k \left(\sqrt{n_k+1}\, |n_1 \dots, n_k+1, \dots n_\infty\rangle\right.$$

$$= \underbrace{\sqrt{n_k+1} \cdot \sqrt{n_k+1}}_{(n_k+1)}\, |n_1 \dots, n_k, \dots n_\infty\rangle$$

$$a_k^+ a_k |n_1 \dots n_\infty\rangle = a_k^+ \left(\sqrt{n_k}\, |n_1 \dots, n_k-1, \dots n_\infty\rangle\right)$$

$$= \underbrace{\sqrt{n_k}\sqrt{n_k}}_{n_k}\, |n_1 \dots, n_k, \dots, n_\infty\rangle$$

$$(a_k a_k^+ - a_k^+ a_k) |n_1 n_2 \dots n_\infty\rangle = |n_1 n_2 \dots n_\infty\rangle$$

Prema tome,

$$[a_k, a_l^+] |n_1 n_2 \dots n_\infty\rangle = \delta_{kl} |n_1 n_2 \dots n_\infty\rangle$$

Proizvoljni vektor stanja je linearna kombinacija stanja $\{|n_1 n_2 \ldots n_\infty\rangle\}$. Prema tome vrijedi

$$[a_k, a_l^+]|\psi\rangle = [a_k, a_l^+] \sum_{\{n_i\}} C_{\{n_i\}} |n_1 n_2 \ldots n_\infty\rangle$$

$$= \sum_{\{n_i\}} C_{\{n_i\}} [a_k, a_l^+] |n_1 n_2 \ldots n_\infty\rangle$$

$$= \delta_{kl} |\psi\rangle$$

Budući je $|\psi\rangle$ proizvoljan vektor stanja

$$[a_k, a_l^+] = \delta_{kl}$$

Provjerimo koliko je $[a_k, a_l]$

$$a_k a_l |n_1 n_2 \ldots n_\infty\rangle = a_k (\sqrt{n_l} |n_1 \ldots, n_l - 1, \ldots n_\infty\rangle)$$

$$= \sqrt{n_l}\sqrt{n_k} |n_1 \ldots, n_k - 1, \ldots, n_l - 1, \ldots n_\infty\rangle$$

$$a_l a_k |n_1 n_2 \ldots n_\infty\rangle = \sqrt{n_l}\sqrt{n_k} |n_1 \ldots, n_k - 1, \ldots, n_l - 1, \ldots n_\infty\rangle$$

$$(a_k a_l - a_l a_k)|n_1 n_2 \ldots n_\infty\rangle = 0$$

Čak i ako je $l = k$,

$$a_k^2 |n_1 n_2 \ldots n_\infty\rangle = \sqrt{n_k}\sqrt{n_k - 1} |n_1 \ldots, n_k - 2, \ldots n_\infty\rangle$$

pa je

$$(a_k^2 - a_k^2)|n_1 n_2 \ldots n_\infty\rangle = 0$$

Prema tome,

$$[a_k, a_l] = 0$$

Ista logika vrijedi i za $[a_k^+, a_e^+] = 0$.

$\boxed{2.}$

(a) Diracov hamiltonijan

$$H = c\vec{\alpha}\cdot\vec{p} + mc^2\beta \quad , \quad \vec{p} \text{ je operator}$$

$$\alpha_j = -\gamma_5 \Sigma_j = -\Sigma_j \gamma_5 \quad (\text{s vježbi})$$

$$\gamma_5 = \begin{pmatrix} 0 & -1\!\!1 \\ -1\!\!1 & 0 \end{pmatrix}$$

$$\Sigma_k = \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}$$

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \quad ; \quad \beta = \begin{pmatrix} 1\!\!1 & 0 \\ 0 & -1\!\!1 \end{pmatrix}$$

$$[H, \vec{\Sigma}\cdot\vec{n}] = [c\,\alpha_j p_j + mc^2\beta, \Sigma_k n_k]$$

j, k su nijemi indeksi; sumira se po njima

$$[\beta, \Sigma_k] = \beta\Sigma_k - \Sigma_k\beta$$
$$= \begin{pmatrix} 1\!\!1 & 0 \\ 0 & -1\!\!1 \end{pmatrix}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} - \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}\begin{pmatrix} 1\!\!1 & 0 \\ 0 & -1\!\!1 \end{pmatrix}$$
$$= \begin{pmatrix} \sigma_k & 0 \\ 0 & -\sigma_k \end{pmatrix} - \begin{pmatrix} \sigma_k & 0 \\ 0 & -\sigma_k \end{pmatrix} = 0$$

komutiraju! Ostaje,

$$[\alpha_j p_j, \Sigma_k n_k] = [\alpha_j, \Sigma_k]\, p_j n_k$$
$$= -\gamma_5 [\Sigma_j, \Sigma_k]\, p_j n_k$$

$$[\Sigma_j, \Sigma_k] = 2i\,\varepsilon_{jkl}\,\Sigma_l$$

Isti uvjet zadovoljavaju $\vec{\sigma}$ matrice! Imamo

$$-c\underbrace{\gamma_5\Sigma_l}_{-\alpha_l}\cdot 2i\,\varepsilon_{jkl}\,p_j n_k = 2i\underbrace{\varepsilon_{jkl}}_{\varepsilon_{klj}}\,c\,\alpha_l\,p_j n_k$$

$$= 2ic\,\vec{n}\cdot(\vec{\alpha}\times\vec{p})$$

Samo ako je $\vec{n}\equiv\hat{\vec{p}}$ ovo je nula. U protivnom, nije!

(b) Operator heliciteta

$$\vec{\Sigma}\cdot\hat{\vec{p}} = \Sigma_k\,\hat{p}_k = \begin{pmatrix}\vec{\sigma}\cdot\hat{\vec{p}} & 0\\ 0 & \vec{\sigma}\cdot\hat{\vec{p}}\end{pmatrix}$$

Tražimo

$$\det\begin{pmatrix}\vec{\sigma}\cdot\hat{\vec{p}}-\lambda\mathbb{1} & 0\\ 0 & \vec{\sigma}\cdot\hat{\vec{p}}-\lambda\mathbb{1}\end{pmatrix}=0$$

Dovoljno je izračunati

$$\det\left(\vec{\sigma}\cdot\hat{\vec{p}}-\lambda\mathbb{1}\right)=0$$

jer je

$$\det\begin{pmatrix}A & 0\\ 0 & B\end{pmatrix} = \det A\cdot\det B$$

za blok matrice. Imamo

$$\det\begin{pmatrix}p_3-\lambda & p_1-ip_2\\ p_1+ip_2 & -p_3-\lambda\end{pmatrix}=0$$

$$-(p_3+\lambda)(p_3-\lambda)-(p_1-ip_2)(p_1+ip_2)=0$$

$$-(p_3^2-\lambda^2)-(p_1^2+p_2^2)=0$$

$$\lambda^2-(p_1^2+p_2^2+p_3^2)=0$$

No, $\vec{p}$ je jedinični vektor za kojeg je

$$p_1^2+p_2^2+p_3^2=1$$

Prema tome,

$$\lambda^2=1 \Rightarrow \lambda=\pm 1$$

(c) Operator rotacije je oblika za česticu spina $\frac{1}{2}$ oko osi $\hat{n}$ za kut ϕ

$$\mathcal{D}(\hat{n},\phi)=\exp\left(-\frac{i\,\vec{\sigma}\cdot\hat{n}}{2}\phi\right)$$

$$=\mathbb{1}\cos\left(\frac{\phi}{2}\right)-i\,\vec{\sigma}\cdot\hat{n}\sin\left(\frac{\phi}{2}\right)$$

Očita generalizacija za 4-dimenzionalni prostor je

$$\mathcal{D}(\hat{n},\phi)=\exp\left(-\frac{i\,\vec{\Sigma}\cdot\hat{n}}{2}\phi\right)$$

$$=\mathbb{1}\cos\left(\frac{\phi}{2}\right)-i\,\vec{\Sigma}\cdot\hat{n}\sin\left(\frac{\phi}{2}\right)$$

Treba izračunati

$$\mathcal{D}^{+}(\hat{n},\phi)(\vec{\Sigma}\cdot\vec{p})\mathcal{D}(\hat{n},\phi)=\Big(\mathbb{1}\cos\frac{\phi}{2}+i\,\underbrace{\vec{\Sigma}}_{=\vec{\Sigma}^{+}}\cdot\hat{n}\sin\left(\frac{\phi}{2}\right)\Big)$$

$$\cdot(\vec{\Sigma}\cdot\hat{p})\left(\mathbb{1}\cos\left(\frac{\phi}{2}\right)-i\,\vec{\Sigma}\cdot\hat{n}\sin\left(\frac{\phi}{2}\right)\right)$$

$$=(\vec{\Sigma}\cdot\hat{p})\cos^2\frac{\phi}{2}-i(\vec{\Sigma}\cdot\hat{p})(\vec{\Sigma}\cdot\hat{n})\cos\left(\frac{\phi}{2}\right)\sin\left(\frac{\phi}{2}\right)$$

$$+i(\vec{\Sigma}\cdot\hat{n})(\vec{\Sigma}\cdot\hat{p})\sin\left(\frac{\phi}{2}\right)\cos\left(\frac{\phi}{2}\right)+(\vec{\Sigma}\cdot\hat{n})(\vec{\Sigma}\cdot\hat{p})(\vec{\Sigma}\cdot\hat{n})\sin^2\left(\frac{\phi}{2}\right) \quad (*)$$

Koristimo identitet koji vrijedi za $\vec{\sigma}$-matrice:

$$(\vec{\Sigma}\cdot\hat{\vec{p}})(\vec{\Sigma}\cdot\hat{\vec{n}}) = \hat{\vec{p}}\cdot\hat{\vec{n}} + i\vec{\Sigma}\cdot(\hat{\vec{p}}\times\hat{\vec{n}})$$

$$(\vec{\Sigma}\cdot\hat{\vec{n}})(\vec{\Sigma}\cdot\hat{\vec{p}}) = \hat{\vec{n}}\cdot\hat{\vec{p}} + i\vec{\Sigma}\cdot(\hat{\vec{n}}\times\hat{\vec{p}})$$

$\hat{\vec{p}}\cdot\hat{\vec{n}} = \hat{\vec{n}}\cdot\hat{\vec{p}}$ to su jedinični vektori, nisu operatori

Prema tome, 2. i 3. član u (*) daju

$$\vec{\Sigma}\cdot(\hat{\vec{p}}\times\hat{\vec{n}})\cos\left(\frac{\phi}{2}\right)\sin\left(\frac{\phi}{2}\right) - \vec{\Sigma}\cdot\underbrace{(\hat{\vec{n}}\times\hat{\vec{p}})}_{-\vec{p}\times\vec{n}}\cos\left(\frac{\phi}{2}\right)\sin\left(\frac{\phi}{2}\right)$$

$$= 2\vec{\Sigma}\cdot(\hat{\vec{p}}\times\hat{\vec{n}})\cos\left(\frac{\phi}{2}\right)\sin\left(\frac{\phi}{2}\right)$$

Računamo 4. član

$$(\vec{\Sigma}\cdot\hat{\vec{n}})\cdot(\vec{\Sigma}\cdot\hat{\vec{p}})(\vec{\Sigma}\cdot\hat{\vec{n}})$$

$$= (\vec{\Sigma}\cdot\hat{\vec{n}})\left(\hat{\vec{p}}\cdot\hat{\vec{n}} + i\vec{\Sigma}\cdot(\hat{\vec{p}}\times\hat{\vec{n}})\right)$$

$$= (\hat{\vec{p}}\cdot\hat{\vec{n}})\,\vec{\Sigma}\cdot\hat{\vec{n}} + i(\vec{\Sigma}\cdot\hat{\vec{n}})\left(\vec{\Sigma}\cdot(\hat{\vec{p}}\times\hat{\vec{n}})\right)$$

No, po istom pravilu:

$$(\vec{\Sigma}\cdot\hat{\vec{n}})\left(\vec{\Sigma}\cdot(\hat{\vec{p}}\times\hat{\vec{n}})\right) = \underbrace{\hat{\vec{n}}\cdot(\hat{\vec{p}}\times\hat{\vec{n}})}_{=0} + i\vec{\Sigma}\cdot\underbrace{\left[\hat{\vec{n}}\times(\hat{\vec{p}}\times\hat{\vec{n}})\right]}_{\hat{\vec{p}}\underbrace{(\hat{\vec{n}}\cdot\hat{\vec{n}})}_{=1} - \hat{\vec{n}}(\hat{\vec{n}}\cdot\hat{\vec{p}})}$$

$$= i\vec{\Sigma}\cdot\hat{\vec{p}} - i(\vec{\Sigma}\cdot\hat{\vec{n}})(\hat{\vec{n}}\cdot\hat{\vec{p}})$$

Prema tome,

$$(\vec{\Sigma}\cdot\vec{n})(\vec{\Sigma}\cdot\vec{p})(\vec{\Sigma}\cdot\vec{n}) = 2(\vec{\Sigma}\cdot\vec{n})\,\vec{p}\cdot\vec{n} - \vec{\Sigma}\cdot\vec{p}$$

Imamo,

$$\begin{aligned} D^{+}(\vec{n},\phi)(\vec{\Sigma}\cdot\vec{p})D(\vec{n},\phi) = (\vec{\Sigma}\cdot\vec{p})\cos^2\frac{\phi}{2} \\ + 2\,\vec{\Sigma}\cdot(\vec{p}\times\vec{n})\cos\frac{\phi}{2}\sin\frac{\phi}{2} \\ + 2(\vec{\Sigma}\cdot\vec{n})(\vec{p}\cdot\vec{n})\sin^2\frac{\phi}{2} - (\vec{\Sigma}\cdot\vec{p})\sin^2\frac{\phi}{2} \end{aligned}$$

$$\cos^2\frac{\phi}{2} - \sin^2\frac{\phi}{2} = \cos\phi$$

$$2\cos\frac{\phi}{2}\sin\frac{\phi}{2} = \sin\phi$$

$$2\sin^2\frac{\phi}{2} = 1-\cos\phi$$

Gornji izraz postaje

$$= (\vec{\Sigma}\cdot\vec{p})\cos\phi + \vec{\Sigma}\cdot(\vec{p}\times\vec{n})\sin\phi + (\vec{\Sigma}\cdot\vec{n})(\vec{p}\cdot\vec{n})(1-\cos\phi)$$

$$= \vec{\Sigma}\cdot\left(\vec{p}\cos\phi + (\vec{p}\times\vec{n})\sin\phi + \vec{n}(\vec{p}\cdot\vec{n})(1-\cos\phi)\right)$$

Izraz u zagradi je Rodriguesova formula za vrtnju vektora $\vec{p}$ oko $\hat{n}$ za kut ϕ (aktivna rotacija)

$$\vec{p}\,' = \vec{p}\cos\phi + (\vec{p}\times\vec{n})\sin\phi + \vec{n}(\vec{p}\cdot\vec{n})(1-\cos\phi)$$

Pokazali smo da vrijedi

$$D^{+}(\vec{n},\phi)(\vec{\Sigma}\cdot\vec{p})D(\vec{n},\phi) = \vec{\Sigma}\cdot\vec{p}\,'$$