The Irreducible Representations of the Lie algebra $\mathfrak{su}(2)$

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Contents

1 Introduction **1**

2 Lie groups and Lie algebras **2**

3 Representations **6**
3.1 Lie algebra $\mathfrak{su}(2)$ 7
3.2 Lie algebra $\mathfrak{so}(3)$ 8

4 SU(2) and SO(3) **9**

5 Irreducible representations of su(2) **15**

6 Physical significance **19**

7 Concluding Remarks **23**

1 Introduction

Symmetry is arguably the most important concept in physics, whether it be rotational symmetry, Lorentz symmetry, or some other symmetry. In many important cases, these symmetries form a *continuous group*, i.e. a group parameterized by real numbers. Mathematically, these groups are often *Lie groups*, smooth manifolds which are also groups. The tangent space at the identity element of the Lie group is a vector space that has a natural *bracket* operation that makes it a *Lie algebra*. The Lie algebra of a Lie group encodes many of the properties of the Lie group and its linearity makes it easy to work with.

In quantum mechanics, symmetry is encoded through a unitary action of the group on the relevant Hilbert space. By Noether's theorem, to every differentiable symmetry generated by local actions (i.e. actions of a Lie group), there corresponds a conserved physical quantity. Observables corresponding to conserved quantities form a representation of the Lie algebra of that group.

For example, invariance of the Hamiltonian operator $\hat{H}$ of a quantum system under rotations means that $\hat{H}$ commutes with the relevant representation of the rotation group and thus also with the associated Lie algebra operators. This commutativity, in turn, implies that the eigenspaces for $\hat{H}$ are invariant under rotations.

At this point, the importance of Lie algebras and their representations should be evident.

In this paper, we investigate the special unitary group SU(2) and the rotation group SO(3). The group SU(2) is particularly important as a double cover of SO(3). We investigate the physical meaning of SO(3) by exploring the Lie algebra $\mathfrak{su}(2)$ of its double cover, SU(2). In particular, we will compute all finite-dimensional irreducible representations of $\mathfrak{su}(2)$ up to isomorphism. The close relationship of $\mathfrak{so}(3)$ with $\mathfrak{su}(2)$ will prove to be especially illuminating.

Basic knowledge of group theory and analytical mechanics is assumed throughout the text.

2 Lie groups and Lie algebras

To prepare the ground, first we have to understand what Lie algebras and their representations are.

Definition 1. *A **Lie algebra** over a field $\mathbb{F}$ ($\mathbf{char}\,\mathbb{F} \neq 2$) is a vector space $\mathfrak{g}$ over $\mathbb{F}$, together with a map (called the **Lie bracket**) $[\,\cdot\,,\cdot\,] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ with the following properties:*
$(\forall a, b \in \mathbb{F})$, $(\forall X, Y, Z \in \mathfrak{g})$

1. *Bilinearity:*
   $[aX + bY, Z] = a[X, Z] + b[Y, Z]$
   $[Z, aX + bY] = a[Z, X] + b[Z, Y]$

2. *Anticommutativity:*
   $[X, Y] = -[Y, X]$

3. *The Jacobi identity:*
   $\big[X, [Y, Z]\big] + \big[Y, [Z, X]\big] + \big[Z, [X, Y]\big] = 0$

*If $\mathbb{F} = \mathbb{R}$ ($\mathbb{F} = \mathbb{C}$), we say that $\mathfrak{g}$ is a **real (complex) Lie algebra**. If $\mathfrak{g}$ is a finite-dimensional space, we say that $\mathfrak{g}$ is a **finite-dimensional Lie algebra**.*

A large class of Lie algebras may be obtained by the following procedure.

Proposition 1. *Let $\mathcal{A}$ be an associative algebra and let $\mathfrak{g}$ be a subspace of $\mathcal{A}$ with the property that for all $X, Y \in \mathfrak{g}$, $XY - YX$ is again in $\mathfrak{g}$. Then the bracket*

$$[X, Y] := XY - YX$$

*makes $\mathfrak{g}$ into a Lie algebra. This bracket is called the **commutator** of X and Y.*

Proof. The proof trivially follows from the definition of the Lie algebra. □

So far, we have discussed Lie algebras as abstract algebraic structures. Let us now turn to Lie algebras of Lie groups. First we define matrix Lie groups.

Definition 2. *Let G be a subgroup of $\mathrm{GL}(n;\mathbb{C})$. Then, G is called a* ***matrix Lie group*** *if for every sequence of matrices in G:*

$$(A_m \longrightarrow A) \implies A \in G \text{ or } A \notin \mathrm{GL}(n;\mathbb{C})$$

In other words, a matrix Lie group is a (topologically) **closed subgroup** of $\mathrm{GL}(n;\mathbb{C})$. Let us list the matrix Lie groups that will be used in this paper. They are also some of the most important examples of Lie groups in general.

Examples of matrix Lie groups

- $\mathrm{GL}(n;\mathbb{R})$ - invertible real matrices
- $\mathrm{GL}(n;\mathbb{C})$ - invertible complex matrices
- $\mathrm{SL}(n;\mathbb{R})$ - invertible real matrices with unit determinant
- $\mathrm{SL}(n;\mathbb{C})$ - invertible complex matrices with unit determinant
- $\mathrm{O}(n)$ - orthogonal real matrices
- $\mathrm{SO}(n)$ - orthogonal real matrices with unit determinant
- $\mathrm{U}(n)$ - unitary matrices
- $\mathrm{SU}(n)$ - unitary matrices with unit determinant
- $\mathrm{Sp}(n) = (\mathrm{Sp}(n;\mathbb{C}) \cap \mathrm{U}(2n)) \subset M_n(\mathbb{H})$ - compact symplectic group (quaternionic unitary matrices)

To associate a Lie algebra to a matrix Lie group, we need the notion of the exponential of a matrix.

Definition 3. *Let $X \in M_n(\mathbb{C})$. The matrix exponential $e^X \equiv \exp(X)$ is defined as* $\exp : M_n(\mathbb{C}) \to \mathrm{GL}(n;\mathbb{C})$

$$e^X = \sum_{m=0}^{\infty} \frac{X^m}{m!}.$$

The matrix exponential shares some but not all of the properties of the exponential of a number. We list some of these properties without proof.

Properties of the matrix exponential

- e^X is a continuous function of X
- the series $\sum_{m=0}^{\infty} \frac{X^m}{m!}$ converges for all $X \in M_n(\mathbb{C})$
- $e^0 = I$
- $\mathbf{det}(e^X) \neq 0$
- $e^{X^T} = (e^X)^T$
- $e^{X^\dagger} = (e^X)^\dagger$
- $(e^X)^{-1} = e^{-X}$
- $e^{(\alpha+\beta)X} = e^{\alpha X} e^{\beta X} \qquad (\forall \alpha, \beta \in \mathbb{C})$
- $XY = YX \implies e^{X+Y} = e^X e^Y = e^Y e^X$
- $C \in \mathrm{GL}(n;\mathbb{C}) \implies e^{CXC^{-1}} = Ce^X C^{-1}$
- $\frac{d}{dt}\Big|_{t=0} e^{tX} = X$
- $e^{X+Y} = \lim_{m\to\infty} (e^{\frac{X}{m}} e^{\frac{Y}{m}})^m \qquad$ (Lie product formula)
- $\mathbf{det}(e^X) = e^{\mathbf{tr}(X)}$

Definition 4. *Let G be a matrix Lie group. The **Lie algebra of the matrix Lie group** G, denoted $\mathfrak{g}$ is defined as:*

$$\mathfrak{g} = \left\{ X \in M_n(\mathbb{C}) \middle| e^{tX} \in G,\, \forall t \in \mathbb{R} \right\},$$

where e^{tX} is the matrix exponential of tX.

In Lie algebras of matrix Lie groups, the matrix commutator takes the role of the Lie bracket.

Definition 5. *Let $A, B \in M_n(\mathbb{C})$. The **commutator** of A and B is given by*

$$[A, B] = AB - BA.$$

Intuitively, if we think of Lie groups as *continuous groups*, elements of its Lie algebra can be interpreted as *differential group elements*, i.e. *group infinitesimals*.

In the general theory of Lie groups, the Lie algebra of a Lie group is geometrically thought of as the tangent space to the identity element and the exponential map is defined as follows.

Definition 6. *Let G be a matrix Lie group and $\mathfrak{g}$ its Lie algebra. The* ***exponential map*** *is a map*

$$\exp : \mathfrak{g} \longrightarrow G,$$

where $exp(X) \equiv e^X$ is the matrix exponential given by ***Definition 3****.*

Every Lie group homomorphism gives rise to a Lie algebra homomorphism in a natural way.

Definition 7. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras over a field $\mathbb{F}$. A linear map $\varphi : \mathfrak{g} \to \mathfrak{h}$ such that*

$$\varphi([X, Y]) = [\varphi(X), \varphi(Y)] \qquad (\forall X, Y \in \mathfrak{g})$$

is called a ***Lie algebra homomorphism****. If φ is a bijective map, we say that φ is a* ***Lie algebra isomorphism****.*

Theorem 1. *Let G and H be matrix Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Suppose $\Phi : G \to H$ is a Lie group homomorphism. Then there exists a unique linear map $\varphi : \mathfrak{g} \to \mathfrak{h}$ such that*

$$\Phi(e^{tX}) = e^{t\varphi(X)}$$

for all $t \in \mathbb{R}$ and $X \in \mathfrak{g}$. This linear map has the following additional properties: $(\forall X, Y \in \mathfrak{g})$, $(\forall A \in G)$

- $\varphi([X, Y]) = [\varphi(X), \varphi(Y)]$
- $\varphi(AXA^{-1}) = \Phi(A)\varphi(X)\Phi(A)^{-1}$
- $\varphi(X) = \frac{d\Phi}{dt}\Big|_{t=0}$

An important fact follows from the preceding theorem.

$$G \cong H \implies \mathfrak{g} \cong \mathfrak{h}$$

Furthermore, every matrix Lie group is a real embedded submanifold of $M_n(\mathbb{C})$ whose dimension is equal to the dimension of its Lie algebra as a real vector space.

3 Representations

Definition 8. *Let G be a matrix Lie group.*
*A **finite-dimensional complex representation** of G is a Lie group homomorphism*

$$\Pi : G \longrightarrow \mathrm{GL}(\mathcal{V}),$$

where $\mathcal{V}$ is a finite-dimensional complex vector space.
*If G is a real matrix Lie group, then a **finite-dimensional real representation** of G is a Lie group homomorphism where $\mathcal{V}$ is a finite-dimensional real vector space.*

Let $\mathfrak{g}$ be a real (or complex) Lie algebra.
*A **finite-dimensional complex representation** of g is a Lie algebra homomorphism*

$$\pi : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathcal{V}),$$

where $\mathcal{V}$ is a finite-dimensional complex vector space.
*If g is a real Lie algebra, then a **finite-dimensional real representation** of $\mathfrak{g}$ is a Lie algebra homomorphism where $\mathcal{V}$ is a finite-dimensional real vector space.*
*If Π or π is an monomorphism, the respective representations are called **faithful**.*

It is fruitful to think of representations as **actions** of groups or Lie algebras on a vector space.

Definition 9. *Let G be a group and let X be a set. An **action** of the group G on X is a map $\cdot : G \times X \to X$, denoted $(g, x) \mapsto g \cdot x$, satisfying $(\forall x \in X), (\forall g, h \in G)$*

1. $e \cdot x = x$
2. $g \cdot (h \cdot x) = (gh) \cdot x$,

where e is the identity element in G.

A representation Π of G on some vector space V gives rise to a *linear action* of G on V, given by:

$$g \cdot \mathbf{v} := \Pi(g)\mathbf{v}$$

Thus, for convenience and clarity, we may use $g \cdot \mathbf{v}$ as an alternative notation to $\Pi(g)\mathbf{v}$.

Naturally, Lie group morphisms are morphisms of differentiable manifolds that preserve the group product. In particular, diffeomorphisms that preserve the group product are Lie group isomorphisms. Similarly, Lie algebra morphisms are vector space morphisms (i.e. linear maps between vector spaces over the same field) that preserve the bracket operation. Morphisms of representation are also defined naturally as follows.

Definition 10. *Let G be a matrix Lie group, let Π be a representation of G acting on the space $\mathcal{V}$, and let Σ be a representation of G acting on the space $\mathcal{W}$. A linear map $\Phi : \mathcal{V} \to \mathcal{W}$ is called a* ***morphism*** *of representations if*

$$\Phi(\Pi(A)\mathbf{v}) = \Sigma(A)\,\Phi(\mathbf{v}) \qquad (\forall A \in G)\,(\forall \mathbf{v} \in \mathcal{V}).$$

The analogous property defines morphisms of representations of a Lie algebra. Naturally, invertible morphisms are called ***isomorphisms*** *and the respective representations are said to be* ***isomorphic*** *or* ***equivalent****.*

One class of representations, namely the irreducible representations is particularly useful and interesting.

Definition 11. *Let π be a finite-dimensional real or complex representation of a Lie algebra $\mathfrak{g}$, acting on a vector space $\mathcal{V}$. A subspace $\mathcal{W} \subseteq \mathcal{V}$ is called* ***invariant*** *if $\pi(\mathcal{W}) \subseteq \mathcal{W}$, i.e. if $\pi(A)\mathbf{w} \in \mathcal{W}$ for all $\mathbf{w} \in \mathcal{W}$ and all $A \in \mathfrak{g}$.*
An invariant subspace $\mathcal{W}$ is called ***nontrivial*** *if $\mathcal{W} \neq \{\mathbf{0}\}$ and $\mathcal{W} \neq \mathcal{V}$.*
A representation with no nontrivial invariant subspaces is called ***irreducible****.*
These terms are defined analogously for representations of Lie groups.

3.1 Lie algebra $\mathfrak{su}(2)$

We are interested in the group SU(2) so let us construct its Lie algebra, $\mathfrak{su}(2)$.

By definition:

$$\mathfrak{su}(2) = \left\{ X \in M_2(\mathbb{C}) \middle| e^{tX} \in \mathrm{SU}(2), \forall t \in \mathbb{R} \right\}$$

Using the defining property of SU(2) in terms of 2×2 matrices, namely

$$\mathrm{SU}(2) = \left\{ X \in M_2(\mathbb{C}) \middle| X^\dagger = X^{-1}, \mathbf{det}(X) = 1 \right\},$$

where $X^\dagger$ denotes the Hermitian conjugate of X (i.e. conjugate transpose of matrix X, explicitly $X^\dagger = (X^T)^*$), we get

$$\mathfrak{su}(2) = \left\{ X \in M_2(\mathbb{C}) \middle| (e^{tX})^\dagger = (e^{tX})^{-1}\,,\, \mathbf{det}(e^{tX}) = 1\,,\, \forall t \in \mathbb{R} \right\}.$$

Since $(e^{tX})^\dagger = e^{(tX)^\dagger} = e^{tX^\dagger}$ and $\mathbf{det}(e^{tX}) = e^{\mathbf{tr}(tX)} = e^{t\,\mathbf{tr}(X)}$, the conditions translate to

$$X^\dagger = -X \;\text{ and }\; \mathbf{tr}(X) = 0.$$

In words, $\mathfrak{su}(2)$, the Lie algebra of the matrix Lie group SU(2) is built upon the set of traceless anti-Hermitian 2×2 complex matrices.

$$\mathfrak{su}(2) = \left\{ X \in M_2(\mathbb{C}) \middle| X^\dagger = -X,\, \mathbf{tr}(X) = 0 \right\}$$

3.2 Lie algebra $\mathfrak{so}(3)$

Similarly, for the Lie algebra of SO(3), by definition:

$$\mathfrak{so}(3) = \left\{ X \in M_3(\mathbb{R}) \middle| e^{tX} \in \mathrm{SO}(3), \forall t \in \mathbb{R} \right\}$$

The defining property of SO(3), in terms of 3×3 matrices is given as

$$\mathrm{SO}(3) = \left\{ X \in M_3(\mathbb{R}) \middle| X^T = X^{-1}\,,\, \mathbf{det}(X) = 1 \right\},$$

thus, its Lie algebra may be written as

$$\mathfrak{so}(3) = \left\{ X \in M_3(\mathbb{R}) \middle| (e^{tX})^T = (e^{tX})^{-1}\,,\, \mathbf{det}(e^{tX}) = 1\,,\, \forall t \in \mathbb{R} \right\}$$

Therefore,

$$\mathfrak{so}(3) = \left\{ X \in M_3(\mathbb{R}) \middle| X^T = -X\,,\, \mathbf{tr}(X) = 0\,,\, \forall t \in \mathbb{R} \right\}.$$

4 SU(2) and SO(3)

Before proceeding to the heart of matter, let us first spell out the relationship between SO(3) and SU(2). To do that, we require the following topological notion.

Definition 12. *Let G be a connected matrix Lie group with Lie algebra $\mathfrak{g}$. A **universal cover** of G is an ordered pair $\left(\widetilde{G}, \Phi\right)$ consisting of a simply connected matrix Lie group $\widetilde{G}$ and a Lie group homomorphism $\Phi : \widetilde{G} \to G$ such that the associated Lie algebra homomorphism $\varphi : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ is an isomorphism of the Lie algebra $\widetilde{\mathfrak{g}}$ of $\widetilde{G}$ with $\mathfrak{g}$. The map Φ is called the **covering map** for $\widetilde{G}$.*
*If Φ is a two-to-one map, we say that $\widetilde{G}$ is a **double cover** of G.*

Although each Lie group has a universal cover that is again a Lie group, the universal cover of a *matrix* Lie group may not be isomorphic to any matrix Lie group. For example, the universal cover of $\mathrm{SL}(2, \mathrm{R})$ is *not* a matrix Lie group. It can be shown, however, that the universal cover of a *compact* Lie group is again a (not necessarily compact) Lie group.

Proposition 2. *The universal cover of the rotation group* SO(3) *is the group* SU(2)*, which is isomorphic to the compact symplectic group* Sp(1)*, the group of unit quaterions. This universal cover is a double cover. In symbols,*

$$\mathrm{SO}(3) \xhookrightarrow{\text{1 to 2}} \mathrm{SU}(2) \cong \mathrm{Sp}(1) = \left\{ \mathbf{x} \in \mathbb{H} \,\middle|\, \|\mathbf{x}\| = 1 \right\}$$

Let Φ denote the Lie group homomorphism $\Phi : \mathrm{SU}(2) \longrightarrow \mathrm{SO}(3)$. Then,

$$\mathbf{Ker}(\Phi) = \{I, -I\} \cong \mathbb{Z}_2.$$

Instead of providing a detailed proof, we demonstrate how to construct such a map. We can write out the elements of SU(2) explicitly as matrices.

$$\mathrm{SU}(2) = \left\{ \begin{bmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{bmatrix} \,\middle|\, |\alpha|^2 + |\beta|^2 = 1,\ \alpha, \beta \in \mathbb{C} \right\}$$

or, separating real from imaginary parts as $\alpha = a + id$ and $\beta = b - ic$.

$$\mathrm{SU}(2) = \left\{ \begin{bmatrix} a + id & -b - ic \\ b - ic & a - id \end{bmatrix} \,\middle|\, a^2 + b^2 + c^2 + d^2 = 1,\ a, b, c, d \in \mathbb{R} \right\}.$$

Now, let us write out Sp(1) explicitly, using $\mathbf{x} = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, where $\mathbf{1}, \mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ are elements of the quaternion group $\mathcal{Q}_8 = \left\langle -\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k} \middle| (-\mathbf{1})^2, \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -\mathbf{1} \right\rangle$.

$$\mathrm{Sp}(1) = \left\{ a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \middle| \, a^2 + b^2 + c^2 + d^2 = 1, \; a, b, c, d \in \mathbb{R} \right\}.$$

The isomorphism between these two groups is now almost obvious, but let us spell it out.

$$\psi : \mathrm{SU}(2) \longrightarrow \mathrm{Sp}(1)$$

$$\begin{bmatrix} a + id & -b - ic \\ b - ic & a - id \end{bmatrix} \overset{\psi}{\longleftrightarrow} (a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k})$$

It is easily seen that ψ is a bijective homomorphism, i.e. an isomorphism.

So far, we have shown (or at least made it plausible) that $\mathrm{SU}(2) \cong \mathrm{Sp}(1)$. Now we use these results to show that $\mathrm{SO}(3) \hookrightarrow \mathrm{SU}(2)$ by constructing a two-to-one map from SU(2) to SO(3).

By letting $\mathbf{1}, \mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ stand for 2×2 complex matrices instead of unit quaternions, we can exploit the $\mathrm{SU}(2) \cong \mathrm{Sp}(1)$ isomorphism and write

$$a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} = \begin{bmatrix} a + id & -b - ic \\ b - ic & a - id \end{bmatrix} \overset{\psi}{\longleftrightarrow} (a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k})$$

which suggests the following basis for SU(2).

$$\mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{i} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \; \mathbf{j} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}, \; \mathbf{k} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

As a side note, these matrices are equal to Pauli matrices, up to a scalar multiplicative constant, but let's leave that on side for now, further results will elucidate that "coincidence".

An arbitrary rotation in three dimensions requires 3 independent real parameters. A quaternion is described by 4 real parameters. Restriction to unit quaternions reduces the number of independent real parameters to 3. SO(3) matrices act on vectors in $\mathbb{R}^3$ by leaving their length and orientation unchanged.

This line of thought begs for a map between vectors in $\mathbb{R}^3$ and quaternions.

$$\mathbb{R}^3 \ni \mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \longmapsto \mathbf{x} \in \mathbb{H}$$

Here, $|\mathbf{v}| = \|\mathbf{x}\|$, i.e. the usual vector norm (its length) is equal to the corresponding quaternion norm. We construct the needed map $\xi : \mathbb{R}^3 \longrightarrow \mathbb{H}$ by the following rule.

$$\xi(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = (0\mathbf{1} + x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

SO(3) matrices act on vectors by matrix multiplication, but the result of multiplying a quaternion of the form described above by a unit quaternion is not necessarily a

quaternion of the same form, i.e. using the notation $\Im(\mathbb{H}) \equiv (\mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}) \subset \mathbb{H}$, we have $\mathrm{Sp}(1)\left(\Im(\mathbb{H})\right) \not\subseteq \Im(\mathbb{H})$.

Instead, we let $\mathrm{Sp}(1)$ act on $\Im(\mathbb{H})$ by

$$\mathbf{x} \longmapsto \mathbf{q}^{-1}\mathbf{x}\mathbf{q} \qquad (\mathbf{x} \in \Im(\mathbb{H})),\ (\mathbf{q} \in \mathrm{Sp}(1))$$

This map, also know as the conjugation of quaternion $\mathbf{x}$ by $\mathbf{q}$ provides us with the desired property of closure of $\Im(\mathbb{H})$ under the action of $\mathrm{Sp}(1)$.

Now comes the crucial part where we put everything together. Having established a connection between $\mathrm{SO}(3)$ and $\mathrm{Sp}(1)$ by investigating its action on three-dimensional vectors, we now proceed by connecting $\mathrm{SO}(3)$ to $\mathrm{SU}(2)$ using the matrix representation of quaternions $\mathbf{1}, \mathbf{i}, \mathbf{j}$, and $\mathbf{k}$.

$$\Im(\mathbb{H}) \ni (0\mathbf{1}+x\mathbf{i}+y\mathbf{j}+z\mathbf{k}) \longmapsto 0\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + x\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} + y\begin{bmatrix}0 & -i\\ -i & 0\end{bmatrix} + z\begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix} \in \mathrm{SU}(2)$$

$$\mathrm{Sp}(1) \ni (a\mathbf{1}+b\mathbf{i}+c\mathbf{j}+d\mathbf{k}) \longmapsto a\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + b\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} + c\begin{bmatrix}0 & -i\\ -i & 0\end{bmatrix} + d\begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix} \in \mathrm{SU}(2)$$

We have now come full circle and we can essentially describe both three-dimensional vectors and their rotations by 2×2 $\mathrm{SU}(2)$ matrices.

An arbitrary vector $\mathbf{v}$ from $\mathbb{R}$ can now be written as follows.

$$\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k} = \begin{bmatrix} iv_z & -v_x - iv_y \\ v_x - iv_y & -iv_z \end{bmatrix}$$

As it is customary in physics, we consider z-axis as the axis of rotation. Furthermore, without loss of generality, we rotate the vector $\mathbf{i}$, the unit vector in the direction of x-axis, i.e.

$$\mathbf{v} = 1\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$$

An arbitrary element of $\mathrm{SO}(3)$, interpreted as a three-dimensional rotation, can be written as the matrix product

$$\mathbf{R} \;=\; \mathbf{R_z}(\alpha)\,\mathbf{R_y}(\beta)\,\mathbf{R_x}(\gamma),$$

where each of the matrices R_z, R_y, and R_x represent rotations about their respective axes by the given angle.

$$\mathbf{R_x}(\gamma) = \begin{bmatrix}1 & 0 & 0\\ 0 & \cos\gamma & \sin\gamma\\ 0 & -\sin\gamma & \cos\gamma\end{bmatrix}$$

$$\mathbf{R_y}(\beta) = \begin{bmatrix}\cos\beta & 0 & -\sin\beta\\ 0 & 1 & 0\\ \sin\beta & 0 & \cos\beta\end{bmatrix}$$

$$\mathbf{R_z}(\alpha) = \begin{bmatrix}\cos\alpha & \sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{bmatrix}$$

Without derivation, let's show that for rotations by θ about the z-axis, we can use the following matrix.

$$R_z(\theta) = \cos(\theta)\mathbf{1} + \sin(\theta)\mathbf{k} = \begin{bmatrix} \cos(\theta) + i\sin(\theta) & 0 \\ 0 & \cos(\theta) - i\sin(\theta) \end{bmatrix} = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$$

The rotated vector is then simply

$$\mathbf{v}' = R_z(\theta)^{-1}\,\mathbf{v}\,R_z(\theta) = \begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} = \begin{bmatrix} 0 & e^{-i2\theta} \\ e^{i2\theta} & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & -\cos(2\theta) + i\sin(2\theta) \\ \cos(2\theta) + i\sin(2\theta) & 0 \end{bmatrix}$$

On the other hand, the rotated vector is represented by the matrix

$$\mathbf{v}' = \begin{bmatrix} iv'_z & -v'_x - iv'_y \\ v'_x - iv'_y & -iv'_z \end{bmatrix}$$

and we know that, using the SO(3) matrix $\mathbf{R}_z(\phi)$,

$$\mathbf{v}' = \mathbf{R}_z(\phi)\,\mathbf{v} = \begin{bmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(\phi) \\ -\sin(\phi) \\ 0 \end{bmatrix}.$$

Therefore,

$$\begin{aligned} &v'_x = \cos(2\theta) = \cos(\phi) \\ &v'_y = -\sin(2\theta) = -\sin(\phi) \\ &v'_z = 0 \\ &\implies R_z(\theta) = \cos(\phi/2)\mathbf{1} + \sin(\phi/2)\mathbf{k}. \end{aligned}$$

Thus, we see that indeed $\mathrm{SO}(3) \hookrightarrow \mathrm{SU}(2) \cong \mathrm{Sp}(1)$ and we see that a rotation of a vector in $\mathbb{R}^3$ by an angle ϕ can be represented by two different unit quaternions or 2×2 matrices, namely one corresponding to $\theta = \frac{\phi}{2}$ and the other to $\theta = \frac{\phi}{2} + \pi$. In other words, we have established a two-to-one map from SU(2) to SO(3).

There is also a shorter and more straightforward way to establish the relationship between SU(2) and SO(3). Although this approach is more rigorous and direct, it lacks the natural intuitive feel of the preceding analysis.

Consider the space $\mathcal{V}$ of all 2×2 traceless anti-Hermitian complex matrices. This is a three-dimensional *real* vector space with the following basis

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix},\ A_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Let us define an inner product on $\mathcal{V}$ by

$$\langle A|B\rangle = \frac{1}{2}\,\mathbf{tr}(AB).$$

A straightforward calculation shows that this product indeed satisfies the inner product axioms, which makes $\mathcal{V}$ an inner product space. By direct computation, it is easily checked that $\{A_1, A_2, A_3\}$ forms an orthonormal basis for $\mathcal{V}$.

Having chosen an orthonormal basis for $\mathcal{V}$, we can identify $\mathcal{V}$ with $\mathbb{R}^3$.

Let $U \in \mathrm{SU}(2)$ and $A \in \mathcal{V}$. Then $UAU^{-1} \in V$. Thus, for each $U \in \mathrm{SU}(2)$, we can define a linear map $\Phi_U : \mathcal{V} \to \mathcal{V}$ by the formula

$$\Phi_U(A) = UAU^{-1}.$$

Moreover, given $U \in \mathrm{SU}(2)$ and $A, B \in \mathcal{V}$, note that

$$\langle \Phi_U(A)|\Phi_U(B)\rangle = \frac{1}{2}\mathbf{tr}(UAU^{-1}UBU^{-1}) = \frac{1}{2}\mathbf{tr}(AB) = \langle A|B\rangle\,.$$

Thus Φ_U is an orthogonal transformation of $\mathcal{V} \cong \mathbb{R}^3$, which can be thought of as an element of $\mathrm{O}(3)$.

The map $U \mapsto \Phi_U$ is an injective map from $\mathrm{SU}(2)$ to $\mathrm{O}(3)$. It is easily checked that this map is a continuous homomorphism, i.e. a Lie group homomorphism of $\mathrm{SU}(2)$ into $\mathrm{O}(3)$.

Matrices in $\mathrm{O}(3)$ have determinant ± 1. Since $\mathrm{SU}(2)$ is connected and the map $U \mapsto \Phi_U$ is continuous, we conclude that Φ_U must actually map into $\mathrm{SO}(3)$. Therefore, $U \mapsto \Phi_U$ is a Lie group homomorphism of $\mathrm{SU}(2)$ into $\mathrm{SO}(3)$.

To show that this map is two-to-one, one begins by observing that for any $U \in \mathrm{SU}(2)$, $\Phi_U = \Phi_{-U}$. For the purposes of this paper, this completes the investigation of the relationship between $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$.

So far, our discussion has been largely informal, with a simple goal in mind, to show the plausibility of the connection between the matrix Lie groups $\mathrm{SO}(3)$ and $\mathrm{SU}(2)$.

$$\mathrm{SO}(3) \xhookrightarrow{1 \text{ to } 2} \mathrm{SU}(2) \cong \mathrm{Sp}(1)$$

Although the steps we have taken to establish the plausibility point in the right direction for a more formal treatment, a rigorous proof is beyond the scope of this paper. Having described the aforementioned groups, we know turn to a detailed treatment of their respective Lie algebras.

It is important to remember that although $\mathfrak{su}(2) \cong \mathfrak{so}(3)$, the corresponding groups are not isomorphic, that is $\mathrm{SU}(2) \ncong \mathrm{SO}(3)$.

There is another way to see this. Since SU(2) (as a manifold) is diffeomorphic to a 3-sphere $\mathcal{S}^3$, it is obviously simply connected (every loop in $\mathcal{S}^3$ can be continuously shrunk to a point in $\mathcal{S}^3$.) On the other hand SO(3) is not simply connected, which can be seen by considering the path that connects the identity element to the point corresponding to a rotation by 2π about any axis. Since those two points are actually the same point, this path is a loop. It is now not difficult to convince ourselves that this loop cannot be continuously shrunk to a point in SO(3).

5 Irreducible representations of su(2)

Theorem 2. *Let $\pi : \mathfrak{so}(3) \to \mathfrak{gl}(\mathcal{V})$ be a finite-dimensional irreducible representation of $\mathfrak{so}(3)$. Let $\{F_1, F_2, F_3\}$ be the following basis for $\mathfrak{so}(3)$.*

$$F_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \; F_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \; F_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We define operators L^+, L^-, and L_3 on $\mathcal{V}$ by

$$\begin{aligned} L^+ &= i\pi(F_1) - \pi(F_2) \\ L^- &= i\pi(F_1) + \pi(F_2) \\ L_3 &= i\pi(F_3). \end{aligned}$$

Let $l = \frac{1}{2}(\mathbf{dim}(\mathcal{V}) - 1)$, so that $\mathbf{dim}(\mathcal{V}) = 2l + 1$. Then, there exists a basis $\mathbf{v}_0, \mathbf{v}_1, \ldots \mathbf{v}_{2l}$ of $\mathcal{V}$ such that

$$L_3\mathbf{v_j} = (l - j)\mathbf{v_j}$$

$$L^-\mathbf{v_j} = \begin{cases} \mathbf{v}_{j+1} & \text{if } j < 2l \\ \mathbf{0} & \text{if } j = 2l \end{cases}$$

$$L^-\mathbf{v_j} = \begin{cases} j(2l + 1 - j)\mathbf{v}_{j-1} & \text{if } j > 0 \\ \mathbf{0} & \text{if } j = 0 \end{cases}$$

Thus, the quantity l completely determines the structure of an irreducible representation of $\mathfrak{so}(3)$. Since $\mathbf{dim}(\mathcal{V})$ is a positive integer, l has to have one of the following values:

$$l = 0, \; \frac{1}{2}, \; 1, \; \frac{3}{2}, \ldots$$

As we have already shown, the Lie algebra $\mathfrak{su}(2)$ can be realised on the set of traceless anti-Hermitian 2×2 complex matrices. On the other hand, the Lie algebra $\mathfrak{so}(3)$ corresponds to antisymmetric 3×3 real (and therefore traceless) matrices, which is easily shown using the orthogonality condition $A^T A = I$ for matrices in O(3).

It is not difficult to construct an isomorphism that clearly shows $\mathfrak{su}(2) \cong \mathfrak{so}(3)$. For example, if we use $\{F_1, F_2, F_3\}$ as the basis for $\mathfrak{so}(3)$ and $\{E_1, E_2, E_3\}$, given by

$$E_1 = \frac{1}{2}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \; E_2 = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \; E_3 = \frac{1}{2}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

as the basis for $\mathfrak{su}(2)$, the map $E_i \mapsto F_i$ clearly preserves the bracket (given by the commutator).

$$[E_i, E_j] = \varepsilon_{ijk}\, E_k$$

Theorem 3. *For any $l = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ there exists an irreducible representation of $\mathfrak{so}(3)$ of dimension $2l+1$, and any two irreducible representations of $\mathfrak{so}(3)$ of dimension $2l+1$ are isomorphic.*

Let us now prove these two main theorems.

Proof of Theorem 2. Since π is a Lie algebra homomorphism, the elements $\pi(F_j)$ satisfy the same commutation relations as F_j themselves. From this we can easily verify the following relations among the operators L^+, L^-, and L_3:

$$\begin{aligned} &[L_3, L^+] = L^+ \\ &[L_3, L^-] = -L^- \\ &[L^+, L^-] = 2L_3. \end{aligned}$$

Now, since we are working over the algebraically closed field $\mathbb{C}$, the operator L_3 has at least one eigenvector $\mathbf{v}$ with eigenvalue λ. Consider, then, $L^+\mathbf{v}$. Using $[L_3, L^+] = L^+$, we compute

$$L_3 L^+\mathbf{v} = (L^+L_3 + L^+)\mathbf{v} = L^+(\lambda\mathbf{v}) + L^+\mathbf{v} = (\lambda+1)L^+\mathbf{v}.$$

Thus, either $L^+\mathbf{v} = 0$ or $L^+\mathbf{v}$ is an eigenvector for L_3 with eigenvalue $\lambda + 1$. We call L^+ the **raising operator**, since it has the effect of raising the eigenvalue of L_3 by 1.

If we apply L^+ repeatedly to $\mathbf{v}$, we obtain eigenvectors for L_3 with eigenvalues increasing by 1 at each step, as long as we do not get the zero vector.

Eventually, though, we must get $\mathbf{0}$, since the operator L_3 has only finitely many eigenvalues.

Thus, there exists $k \geq 0$ such that $(L^+)^k\mathbf{v} \neq \mathbf{0}$ but $(L^+)^{k+1}\mathbf{v} = \mathbf{0}$.

By applying the above formula for $L_3L^+\mathbf{v}$, we see that $(L^+)^k\mathbf{v}$ is an eigenvector for L_3 with eigenvalue $\lambda + k$.

Let us now introduce the notation $\mathbf{v}_0 := (L^+)^k \mathbf{v}$ and $\mu = \lambda + k$. Then $\mathbf{v}_0$ is a nonzero vector with $L^+\mathbf{v}_0 = \mathbf{0}$ and $L_3\mathbf{v}_0 = \mu\mathbf{v}_0$. We now forget about the original vector $\mathbf{v}$ with eigenvalue λ and consider only $\mathbf{v}_0$ and μ. Define vectors $\mathbf{v}_j$ by

$$\mathbf{v}_j = (L^-)^j \mathbf{v}_0, \qquad j = 0, 1, 2, \ldots$$

Arguing as in the case of $L_3L^+\mathbf{v}$, but using $[L_3, L^-] = -L^-$ instead of $[L_3, L^+] = L^+$, we see that L^- has the effect of either lowering the eigenvalue of L_3 by 1 or of giving the zero vector. Thus $L_3\mathbf{v}_j = (\mu - j)\mathbf{v}_j$.

Next, we claim that for $j \geq 1$ we have

$$L^+\mathbf{v}_j = j(2\mu + 1 - j)\mathbf{v}_j,$$

which is easily proved by induction on j. Since, again, L_3 has only finitely many eigenvectors, $\mathbf{v}_j$ must eventually be zero.

Thus, there exists some $N \geq 0$ such that $\mathbf{v}_N \neq \mathbf{0}$ but $\mathbf{v}_{N+1} = \mathbf{0}$. Since $\mathbf{v}_{N+1} = \mathbf{0}$, applying the relation that we have just proved using induction, with $j = N$, gives

$$\mathbf{0} = L^+\mathbf{v}_{N+1} = (N + 1)(2\mu - N)\mathbf{v}_N.$$

Since $\mathbf{v}_N \neq 0$ and $N + 1 > 0$, we must have $(2\mu - N) = 0$. This means that μ must equal $N/2$.

Letting $l = N/2$ and putting $\mu = l$, we have the formulas claimed in the theorem.

$$L_3\mathbf{v_j} = (l - j)\mathbf{v_j}$$

$$L^-\mathbf{v_j} = \begin{cases} \mathbf{v}_{j+1} & \text{if } j < 2l \\ \mathbf{0} & \text{if } j = 2l \end{cases}$$

$$L^-\mathbf{v_j} = \begin{cases} j(2l + 1 - j)\mathbf{v}_{j-1} & \text{if } j > 0 \\ \mathbf{0} & \text{if } j = 0 \end{cases}$$

Meanwhile, since $\{\mathbf{v}_j\}$ are eigenvectors for L_3 with distinct eigenvalues, they are automatically linearly independent.

Furthermore, the span of $\{\mathbf{v}_j\}$ is invariant under L^+, L^-, and L_3, hence under all of $\mathfrak{so}(3)$.

Since $\mathcal{V}$ is assumed to be irreducible, the span of $\{\mathbf{v}_j\}$ must be all of $\mathcal{V}$.

Thus, $\{\mathbf{v}_j\}$ forms a basis for $\mathcal{V}$. The dimension of $\mathcal{V}$ is therefore equal to the number of vectors in $\{\mathbf{v}_j\}$, which is $N + 1 = 2l + 1$. □

Proof of Theorem 3. We construct $\mathcal{V}$ simply by *defining* a space $\mathcal{V}$ with basis $\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_{2l}$ and defining the action of $\mathfrak{so}(3)$ by

$$
\begin{aligned}
L_3\mathbf{u_j} &= (l-j)\mathbf{u_j} \\
L^-\mathbf{u_j} &= \begin{cases} \mathbf{u}_{j+1} & \text{if } j < 2l \\ \mathbf{0} & \text{if } j = 2l \end{cases} \\
L^-\mathbf{u_j} &= \begin{cases} j(2l+1-j)\mathbf{u}_{j-1} & \text{if } j > 0 \\ \mathbf{0} & \text{if } j = 0 \end{cases}
\end{aligned}
$$

A straightforward calculation shows that L^+, L^-, and L_3, defined in this way, have the correct commutation relations, so that $\mathcal{V}$ is indeed a representation of $\mathfrak{so}(3)$. It remains to show that $\mathcal{V}$ is irreducible. Suppose that $\mathcal{W}$ is an invariant subspace of $\mathcal{V}$ and that $\mathcal{W} \neq \{\mathbf{0}\}$. We need to show that $\mathcal{W} = \mathcal{V}$. To this end, suppose that $\mathbf{w}$ is some nonzero element of $\mathcal{W}$, which we can decompose as

$$\mathbf{w} = \sum_{j=0}^{2l} a_j \mathbf{u}_j.$$

Let j_0 be the largest index for which a_j is nonzero. According to the formula for L^+, applying L^+ to any of the vectors $\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_{2l}$ gives a *nonzero* multiple of the previous element in our chain. Thus, $(L^+)^{j_0}\mathbf{w}$ will be a nonzero multiple of $\mathbf{u}_0$. Since $\mathcal{W}$ is invariant, this means that $\mathbf{u}_0 \in \mathcal{W}$. But then by applying L^- repeatedly, we see that $\mathbf{u}_j$ belongs to $\mathcal{W}$ for each j. Therefore, $\mathcal{W} = \mathcal{V}$.

Theorem 2 tells us that any irreducible representation of so(3) of dimension $2l+1$ has a basis $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{2l}$ such that:

$$
\begin{aligned}
L_3\mathbf{v_j} &= (l-j)\mathbf{v_j} \\
L^-\mathbf{v_j} &= \begin{cases} \mathbf{v}_{j+1} & \text{if } j < 2l \\ \mathbf{0} & \text{if } j = 2l \end{cases} \\
L^-\mathbf{v_j} &= \begin{cases} j(2l+1-j)\mathbf{v}_{j-1} & \text{if } j > 0 \\ \mathbf{0} & \text{if } j = 0 \end{cases}
\end{aligned}
$$

We can then construct an isomorphism by mapping this basis in one space to the corresponding basis in the other space, i.e. $\{\mathbf{u}_j\} \leftrightarrow \{\mathbf{v}_j\}$. □

6 Physical significance

Symmetry is arguably the most important concept in fundamental theoretical physics. Symmetries of physical systems are often continuous and therefore mathematically realised as Lie groups. A physical system often has more than one type of symmetry. Consequently, physical symmetries are often constructed as products of groups.

Group theory reminder: Products of groups

1. **Direct product** of two groups G and H:
   - defined on the set: $G \times H$
   - group product given by:
   $$(g_1, h_1)(g_2, h_2) = (g_1 h_1, g_2 h_2)$$
   - notation: $G \times H$
2. **Semidirect product** of groups H and K:
   - defined on the set: $H \times K$
   - group product given by:
   $$(h_1, k_1)(h_2, k_2) = (h_1 \varphi\left[k_1\right](h_2), k_1 k_2),$$
   where $\varphi : K \to \mathrm{Aut}(H)$ is a group homomorphism.
   - notation: $H \rtimes K$

Tthe physical meaning of some symmetry being a direct product is most easily understood in the case of $E^+(3)$, the Euclidean group of rigid motions in a three-dimensional space $\mathbb{R}^3$. This group is a semidirect product of the translation group T (the set of maps on $\mathbb{R}^3$ of the form $T_{\mathbf{x}}(\mathbf{y}) = \mathbf{x} + \mathbf{y}$) and the rotation group $\mathrm{SO}(3)$.

$$E^+(3) = T \rtimes \mathrm{SO}(3)$$

$$\varphi : \mathrm{SO}(3) \to \mathrm{Aut}(T)$$

A rigid motion can be thought of as a rotation followed by a translation. Consequently, every element of $E^+(3)$ can be written as a pair $\{T_{\mathbf{x}}, R\}$ with $\mathbf{x} \in \mathbb{R}^3$ and $R \in \mathrm{SO}(3)$. Note that for every $\mathbf{y} \in \mathbb{R}^3$,

$$\{T_{\mathbf{x}}, R\}\, \mathbf{y} = R\mathbf{y} + \mathbf{x},$$

$$\{T_{\mathbf{x}_1}, R_1\}\{T_{\mathbf{x}_2}, R_2\}\mathbf{y} = R_1R_2\mathbf{y} + (\mathbf{x}_1 + R_1\mathbf{x}_2).$$

Thus, the group product is

$$\{T_{\mathbf{x}_1}, R_1\}\{T_{\mathbf{x}_2}, R_2\} = \{T_{\mathbf{x}_1}T_{R_1\mathbf{x}_2}, R_1R_2\} = \{T_{\mathbf{x}_1}\,\varphi[R_1](T_{\mathbf{x}_2}), R_1R_2\},$$

which is exactly how we define the group product for a semidirect product of groups. Note that we arrived at this result by postulating the form of the symmetry operation on the basis of our physical intuition about rigid motions in three dimensions.

In non-relativistic classical mechanics, spacetime is realised as the Euclidean space $\mathbb{E}^4$ together with a linear map $t : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ called **time**, and **distance** as the usual scalar product on $\mathbb{R}^3$, which form the **Galilean space** $\mathbb{G}^4$ In general-relativistic classical mechanics, spacetime is realised as a four-dimensional pseudo-Riemannian manifold $\mathcal{M}(1,3)$, while in special-relativistic mechanics (a.k.a in flat spacetime) it is the **Minkowski spacetime** $\mathbb{M}^4$, a pseudo-Euclidean space with metric signature $(1,3)$.

In all of these cases, **three-dimensional spatial rotations** act as isometries of spacetime. Isometries of $\mathbb{G}^4$ form the **Galilean group** $\mathrm{SGal}(3) \cong (\mathbb{R}^3 \times \mathbb{R}) \rtimes (\mathbb{R}^3 \rtimes \mathrm{SO}(3))$, while isometries of $\mathbb{M}^4$ form the **Poincaré group** $\mathrm{P}(1,3) \cong (\mathbb{R}^3 \times \mathbb{R}) \rtimes \mathrm{SO}^+(1,3)$, where $\mathrm{SO}^+(1,3)$ stands for the **proper ortochronous Lorentz group**, the group formed by Lorentz transformations that preserve orientation and the direction of time. The fact that spatial rotations form a subset of these isometries is expressed as

$$\mathrm{SO}(3) < \mathrm{SGal}(3)$$
$$\mathrm{SO}(3) < \mathrm{P}(1,3),$$

i.e. the group of three-dimensional space rotations is a subgroup of both Galilean and Poincaré groups. Naturally, since non-relativistic classical mechanics arises as the limit of relativistic mechanics where $c \to \infty$, one can obtain the Galilean group through the same limit from the Poincaré group by a procedure known as group contraction, whose details lie beyond the scope of this paper.

As hinted at in the introduction, an important example of representations in quantum theory arises from the time-independent Schrödinger equation in $\mathbb{R}^3$,

$$\hat{H}\Psi = E\Psi,$$

where $E \in \mathbb{R}$ is a fixed constant, i.e. energy eigenvalue corresponding to eigenstate Ψ. Since the Hamiltonian encodes all of the properties of a physical system, rotational invariance of a system corresponds to rotational invariance of $\hat{H}$. Thus, the space of solutions of the time-independent Schrödinger equation is invariant under rotations. This does textitnot mean that an individual solution is rotationally invariant, but if $\hat{H}$ is rotationally invariant, rotating a solution of the

time-independent Schrödinger equation produces another solution to the equation. Even if the quantum Hilbert space is infinite-dimensional, the solution spaces of the time-independent Schrödinger equation are typically finite-dimensional and constitute **finite-dimensional representations of** SO(3)**.** Therefore, understanding the finite-dimensional representations of SO(3) provides a deeper understanding of rotationally invariant physical systems.

It has been argued that the most fundamental law of physics is the law of *conservation of information*. In quantum mechanics, it is realised as conservation of *distinction* of quantum states. These distinctions are ultimately distinctions in measurements, which in turn translate to distinctions in expectation values of observables, i.e. linear combinations of eigenvalues of Hermitian operators corresponding to the probabilistic expected value of a physically observable quantity. Thus, measurable distinction of quantum states means orthogonality of these states. Therefore, the inner product on the quantum Hilbert space must be preserved.

Group actions that preserve the inner product as well as the linear structure of the Hilbert space are **unitary** actions.

Definition 13. *Let $\mathcal{H}$ be a finite-dimensional Hilbert space over $\mathbb{C}$ and let $\mathrm{U}(\mathcal{H})$ denote the group of invertible linear transformations of $\mathcal{H}$ that preserve the inner product. A finite-dimensional* ***unitary representation*** *of a matrix Lie group G is a continuous homomorphism of $\Pi : G \to \mathrm{U}(\mathcal{V})$.*

Two unit vectors in the quantum Hilbert space that differ by a multiplication by a constant are considered to represent the same physical state. Thus, an operator of the form $e^{i\theta}I$, with $\theta \in \mathbb{R}$ acts as the identity at the level of physical states. Thus, it is natural to factor out the group consisting of those transformations (which is evidently isomorphic to the circle group $\mathrm{U}(1)$) which results in the following algebraic structure.

Definition 14. *Let $\mathcal{H}$ be a finite-dimensional Hilbert space over $\mathbb{C}$. The* ***projective unitary group*** *over $\mathcal{H}$ is the quotient group*

$$\mathrm{PU}(\mathcal{H}) = {}^{\mathrm{U}(\mathcal{H})}\!/_{\mathrm{U}(1)}.$$

This redundant freedom may look like a mathematical artefact, an annoyance we simply cannot get rid of, but it is actually a non-trivial fact of the nature of physical laws. Their importance is evident in the Lagrangian formalism of quantum field theory where so-called **gauge theories** arise naturally from these symmetries. The term **gauge** refers precisely to these redundant degrees of freedom in the field Lagrangian. A particular choice of a gauge corresponds, for example, to multiplying each of the eigenstates by $e^{i\theta_0}$ (for some fixed $\theta_0 \in \mathbb{R}$). The transformations between possible gauges, called **gauge transformations**, form a Lie group referred to as

the **gauge group** of the theory. The requirement that the theory be gauge invariant, produces vector fields, called the **gauge fields** that have to be included in the Lagrangian to ensure the symmetry of the theory. It is precisely these fields that give rise to particles known as the **gauge bosons**. For example, the gauge group of the Standard Model of particle physics is $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$, where the Abelian group U(1) corresponds to *electromagnetism*, but is usually viewed in the context of a unified theory corresponding to the non-Abelian gauge group $\mathrm{SU}(2) \times \mathrm{U}(1)$, i.e. the *electroweak interaction* with $W^{\pm}$ and Z^0 bosons and the photon as gauge bosons. The non-Abelian group SU(3) corresponds to *strong interaction* with gluons as gauge bosons.

Using the results from our main theorems, the relationship to quantum particles may be shown to be given by the following definition.

Definition 15. *If $(\pi, \mathcal{V})$ is an irreducible finite-dimensional representation of $\mathfrak{so}(3)$, then the **spin** of $(\pi, \mathcal{V})$ is the largest eigenvalue of the operator L_3. Equivalently, l is the unique number such that $\mathbf{dim}(\mathcal{V}) = 2l + 1$. A particle for which the spin is an integer is called a **boson**, and a particle for which the spin is a half-integer is called a **fermion**.*

7 Concluding Remarks

Classically, angular momentum may be thought of as the Hamiltonian generator of rotations, which is particularly obvious in Poisson's formulation of classical mechanics (for the sake of brevity, we assume that the reader is already familiar with analytical mechanics). In this formulation, that relationship is surprisingly simple.

$$\{\hat{H}, \mathbf{J}\} = 0 \implies \{\mathbf{J}, \hat{H}\} = 0$$

Mathematically, this is obviously trivial. Nevertheless, it reflects a profound result, i.e. the rotational invariance of the Hamiltonian $\hat{H}$ guarantees the conservation of angular momentum $\mathbf{J}$. The two sides of this relationship can be read as follows.

The physical system described by $\hat{H}$ is invariant under rotation, which is the transformation generated by $\mathbf{J}$.

The angular momentum vector $\mathbf{J}$ is invariant under time translation, which is the transformation generated by $\hat{H}$.

Quantum mechanically, angular momentum is still the *generator* of rotations, namely the infinitesimal generator of a one-parameter group of unitary rotation operators. It is easily seen that the Poisson bracket is related to functions on the phase space in a way that is similar to how the Lie bracket is related to operators on the quantum Hilbert space. An elegant relationship connects quantum and classical mechanics.

$$\left[\hat{F}, \hat{G}\right] = i\hbar\{F, G\} + \mathcal{O}(\hbar^2)$$

Although this relationship is far from straightforward and its subtleties and technicalities (e.g. Weyl quantization) are not completely understood, it allows us to establish the plausibility of the analogy between the role of symmetry in classical mechanics and in quantum mechanics. Namely, we can express the rotational invariance of the Hamiltonian using the commutator of corresponding matrix operators.

$$\left[\hat{H}, \mathbf{J}\right] = 0 \implies \left[\mathbf{J}, \hat{H}\right] = 0$$

This means that if the Hamiltonian $\hat{H}$ is invariant under rotations, it commutes with the angular momentum operators, in which case, the angular momentum operators are constants of motion in the quantum mechanical sense. If $\hat{H}$ commutes with each component of the angular momentum, each eigenspace of $\hat{H}$ is invariant under the

angular momentum operators. We now state the main conclusion, which justifies our investigation of $\mathfrak{so}(3)$ and consequently of $\mathfrak{su}(2)$.

The eigenspace of $\hat{H}$ (i.e. the solution space to $\hat{H}\Psi = \lambda\Psi$ for a given λ) is invariant under the angular momentum operators.
Thus, the eigenspace constitutes a **representation** of the Lie algebra $\mathfrak{so}(3)$.
The classification of the finite-dimensional irreducible representations of $\mathfrak{so}(3)$ allows us to determine completely the angular dependence of a solution $\Psi(\mathbf{r})$, leaving only the radial dependence of Ψ to be determined.
This has the effect of reducing the number of independent variables from three to one (i.e. $\Psi(r, \phi, \theta) = \psi(r)$), thereby reducing the problem to solving an **ordinary** differential equation.

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