ODJEL ZA FIZIKU SVEUČILIŠTE U RIJECI

Preddiplomski studij Fizika

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1. U ovom zadatku, vrijednosti spina izražene su u jedinicama od $\hbar$. Promotrite sustav koji se sastoji od dvije neinteragirajuće čestice: za česticu 1 spin je 1, a čestica 2 ima spin 1/2. Mjerenje je pokazalo sljedeće rezultate:

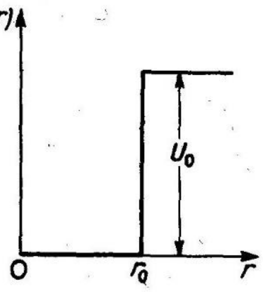
$$s_{1z} = -1;\ s_{2y} = \frac{1}{2}$$

Ovdje su $\mathbf{S}_1$, $\mathbf{S}_2$ operatori spina čestice 1 i 2, a S_{1z} , S_{2y} operatori projekcije spina na os z za prvu česticu i os y za drugu česticu. Njihove svojstvene vrijednosti su s_{1z} , s_{2y} . Kolika je vjerojatnost da za ukupni spin sustava izmjerimo vrijednost $s = 1/2$?

2. Čestica se giba u sferno-simetričnom potencijalu prikazanom na slici. U području $r \leq 0$ valna funkcija je jednaka nuli.
(a) Riješite Schrödingerovu jednadžbu za ovaj problem i nađite valne funkcije ako je energija $E < U_0$.
(b) Nađite jednadžbu iz koje se mogu izračunati energije. Jednadžba se ne može riješiti analitički pa je dovoljno da je napišete u što jasnijem obliku.
(c) Napišite uvjet dobiven pod (b) za $l = 0$. Možete li grafički naći rješenja ove jednadžbe?
***Uputa*:** kovergentna rješenja jednadžbe

$$x^2 y'' + 2xy' - \left[\lambda^2 x^2 + l\left(l+1\right)\right] y = 0$$

nazivaju se modificirane sferne Besselove funkcije druge vrste, $y = k_l(\lambda x)$. U dodatku se nalaze neke osnovne informacije o ovim funkcijama.



3. (a) Provjerite zadovoljava li vektorski potencijal

$$\mathbf{A} = -\frac{1}{2}\mathbf{r}\times\mathbf{B}$$

uvjet $\nabla \cdot \mathbf{A} = 0$, gdje je $\mathbf{B}$ konstantno magnetsko polje.
(b) Neka je $\mathbf{A}$ vektorski potencijal zadan pod (a). Pokažite da je član

$$-\frac{q}{m}\mathbf{A}\cdot\mathbf{p}$$

koji se pojavljuje u hamilotonijanu za česticu u EM polju jednak izrazu za interakciju magnetskog momenta μ_L zbog orbitalnog angularnog momenta s magnetskim poljem $\mathbf{B}$

$$-\boldsymbol{\mu}_L\cdot\mathbf{B} = -\frac{q}{2m}\mathbf{L}\cdot\mathbf{B} = -\frac{q}{2m}\mathbf{B}\cdot\mathbf{L}$$

gdje je q naboj, a m masa čestice.
(c) (NIJE OBAVEZNO) Je li tvrdnja pod (b) točna ako je $\mathbf{B} = \mathbf{B}(\mathbf{r}, t)$? Pretpostavimo da smo našli potencijal za kojeg je $\nabla \cdot \mathbf{A} = 0$. Je li operator $-\boldsymbol{\mu}_L \cdot \mathbf{B}$ uopće hermitski?

4. (a) Nađite svojstvene funkcije operatora $\mathbf{L}^2$ i L_z u impulsnoj reprezentciji, dakle, kao valne funkcije iz impulsnog prostora.
(b) Pokažite da je prosječna vrijednost impulsa

$$\langle\mathbf{p}\rangle = 0$$

u stanju kojem ima oštro definirane svojstvene vrijednosti $\mathbf{L}^2$ i L_z , odnosno, l i m.

Uputa**:** pod (a), prisjetite se da su u impulsnoj reprezentaciji, operator položaja **r** i impulsa **p** jednaki

$$\langle \mathbf{p}' | \mathbf{r} | \psi \rangle = -\frac{\hbar}{i} \nabla_{\mathbf{p}'} \psi(\mathbf{p}')$$

$$\langle \mathbf{p}' | \mathbf{p} | \psi \rangle = \mathbf{p}' \psi(\mathbf{p}')$$

gdje je $\nabla_{\mathbf{p}'} = (\partial/\partial p'_x , \partial/\partial p'_y , \partial/\partial p'_z)$.

44. Clebsch-Gordan Coefficients, Spherical Harmonics, and *d* Functions

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for −8/15 read $-\sqrt{8/15}$.

Notation: columns headed J, M, …; rows labelled m_1, m_2; entries are the Coefficients.

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

1/2×1/2

m_1	m_2	M	$J=1$	$J=0$
+1/2	+1/2	+1	1	
+1/2	−1/2	0	1/2	1/2
−1/2	+1/2	0	1/2	−1/2
−1/2	−1/2	−1	1	

1×1/2

m_1	m_2	M	$J=3/2$	$J=1/2$
+1	+1/2	+3/2	1	
+1	−1/2	+1/2	1/3	2/3
0	+1/2	+1/2	2/3	−1/3
0	−1/2	−1/2	2/3	1/3
−1	+1/2	−1/2	1/3	−2/3
−1	−1/2	−3/2	1	

2×1

m_1	m_2	M	$J=3$	$J=2$	$J=1$
+2	+1	+3	1		
+2	0	+2	1/3	2/3	
+1	+1	+2	2/3	−1/3	
+2	−1	+1	1/15	1/3	3/5
+1	0	+1	8/15	1/6	−3/10
0	+1	+1	2/5	−1/2	1/10
+1	−1	0	1/5	1/2	3/10
0	0	0	3/5	0	−2/5
−1	+1	0	1/5	−1/2	3/10
0	−1	−1	2/5	1/2	1/10
−1	0	−1	8/15	−1/6	−3/10
−2	+1	−1	1/15	−1/3	3/5
−1	−1	−2	2/3	1/3	
−2	0	−2	1/3	−2/3	
−2	−1	−3	1		

1×1

m_1	m_2	M	$J=2$	$J=1$	$J=0$
+1	+1	+2	1		
+1	0	+1	1/2	1/2	
0	+1	+1	1/2	−1/2	
+1	−1	0	1/6	1/2	1/3
0	0	0	2/3	0	−1/3
−1	+1	0	1/6	−1/2	1/3
0	−1	−1	1/2	1/2	
−1	0	−1	1/2	−1/2	
−1	−1	−2	1		

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^{\,\ell}_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

2×1/2

m_1	m_2	M	$J=5/2$	$J=3/2$
+2	+1/2	+5/2	1	
+2	−1/2	+3/2	1/5	4/5
+1	+1/2	+3/2	4/5	−1/5
+1	−1/2	+1/2	2/5	3/5
0	+1/2	+1/2	3/5	−2/5
0	−1/2	−1/2	3/5	2/5
−1	+1/2	−1/2	2/5	−3/5
−1	−1/2	−3/2	4/5	1/5
−2	+1/2	−3/2	1/5	−4/5
−2	−1/2	−5/2	1	

3/2×1/2

m_1	m_2	M	$J=2$	$J=1$
+3/2	+1/2	+2	1	
+3/2	−1/2	+1	1/4	3/4
+1/2	+1/2	+1	3/4	−1/4
+1/2	−1/2	0	1/2	1/2
−1/2	+1/2	0	1/2	−1/2
−1/2	−1/2	−1	3/4	1/4
−3/2	+1/2	−1	1/4	−3/4
−3/2	−1/2	−2	1	

3/2×1

m_1	m_2	M	$J=5/2$	$J=3/2$	$J=1/2$
+3/2	+1	+5/2	1		
+3/2	0	+3/2	2/5	3/5	
+1/2	+1	+3/2	3/5	−2/5	
+3/2	−1	+1/2	1/10	2/5	1/2
+1/2	0	+1/2	3/5	1/15	−1/3
−1/2	+1	+1/2	3/10	−8/15	1/6
+1/2	−1	−1/2	3/10	8/15	1/6
−1/2	0	−1/2	3/5	−1/15	−1/3
−3/2	+1	−1/2	1/10	−2/5	1/2
−1/2	−1	−3/2	3/5	2/5	
−3/2	0	−3/2	2/5	−3/5	
−3/2	−1	−5/2	1		

$$d^{\,j}_{m',m} = (-1)^{m-m'} d^{\,j}_{m,m'} = d^{\,j}_{-m,-m'}$$

$$d^{\,1}_{0,0} = \cos\theta \qquad d^{\,1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{\,1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2}$$

$$d^{\,1}_{1,1} = \frac{1+\cos\theta}{2} \qquad d^{\,1}_{1,0} = -\frac{\sin\theta}{\sqrt{2}} \qquad d^{\,1}_{1,-1} = \frac{1-\cos\theta}{2}$$

2×3/2

m_1	m_2	M	$J=7/2$	$J=5/2$	$J=3/2$	$J=1/2$
+2	+3/2	+7/2	1			
+2	+1/2	+5/2	3/7	4/7		
+1	+3/2	+5/2	4/7	−3/7		
+2	−1/2	+3/2	1/7	16/35	2/5	
+1	+1/2	+3/2	4/7	1/35	−2/5	
0	+3/2	+3/2	2/7	−18/35	1/5	
+2	−3/2	+1/2	1/35	6/35	2/5	2/5
+1	−1/2	+1/2	12/35	5/14	0	−3/10
0	+1/2	+1/2	18/35	−3/35	−1/5	1/5
−1	+3/2	+1/2	4/35	−27/70	2/5	−1/10
+1	−3/2	−1/2	4/35	27/70	2/5	1/10
0	−1/2	−1/2	18/35	3/35	−1/5	−1/5
−1	+1/2	−1/2	12/35	−5/14	0	3/10
−2	+3/2	−1/2	1/35	−6/35	2/5	−2/5
0	−3/2	−3/2	2/7	18/35	1/5	
−1	−1/2	−3/2	4/7	−1/35	−2/5	
−2	+1/2	−3/2	1/7	−16/35	2/5	
−1	−3/2	−5/2	4/7	3/7		
−2	−1/2	−5/2	3/7	−4/7		
−2	−3/2	−7/2	1			

3/2×3/2

m_1	m_2	M	$J=3$	$J=2$	$J=1$	$J=0$
+3/2	+3/2	+3	1			
+3/2	+1/2	+2	1/2	1/2		
+1/2	+3/2	+2	1/2	−1/2		
+3/2	−1/2	+1	1/5	1/2	3/10	
+1/2	+1/2	+1	3/5	0	−2/5	
−1/2	+3/2	+1	1/5	−1/2	3/10	
+3/2	−3/2	0	1/20	1/4	9/20	1/4
+1/2	−1/2	0	9/20	1/4	−1/20	−1/4
−1/2	+1/2	0	9/20	−1/4	−1/20	1/4
−3/2	+3/2	0	1/20	−1/4	9/20	−1/4
+1/2	−3/2	−1	1/5	1/2	3/10	
−1/2	−1/2	−1	3/5	0	−2/5	
−3/2	+1/2	−1	1/5	−1/2	3/10	
−1/2	−3/2	−2	1/2	1/2		
−3/2	−1/2	−2	1/2	−1/2		
−3/2	−3/2	−3	1			

2×2

m_1	m_2	M	$J=4$	$J=3$	$J=2$	$J=1$	$J=0$
+2	+2	+4	1				
+2	+1	+3	1/2	1/2			
+1	+2	+3	1/2	−1/2			
+2	0	+2	3/14	1/2	2/7		
+1	+1	+2	4/7	0	−3/7		
0	+2	+2	3/14	−1/2	2/7		
+2	−1	+1	1/14	3/10	3/7	1/5	
+1	0	+1	3/7	1/5	−1/14	−3/10	
0	+1	+1	3/7	−1/5	−1/14	3/10	
−1	+2	+1	1/14	−3/10	3/7	−1/5	
+2	−2	0	1/70	1/10	2/7	2/5	1/5
+1	−1	0	8/35	2/5	1/14	−1/10	−1/5
0	0	0	18/35	0	−2/7	0	1/5
−1	+1	0	8/35	−2/5	1/14	1/10	−1/5
−2	+2	0	1/70	−1/10	2/7	−2/5	1/5
+1	−2	−1	1/14	3/10	3/7	1/5	
0	−1	−1	3/7	1/5	−1/14	−3/10	
−1	0	−1	3/7	−1/5	−1/14	3/10	
−2	+1	−1	1/14	−3/10	3/7	−1/5	
0	−2	−2	3/14	1/2	2/7		
−1	−1	−2	4/7	0	−3/7		
−2	0	−2	3/14	−1/2	2/7		
−1	−2	−3	1/2	1/2			
−2	−1	−3	1/2	−1/2			
−2	−2	−4	1				

$$d^{\,3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,1/2} = -\sqrt{3}\,\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-1/2} = \sqrt{3}\,\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^{\,2}_{2,2} = \left(\frac{1+\cos\theta}{2}\right)^2$$

$$d^{\,2}_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^{\,2}_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,-2} = \left(\frac{1-\cos\theta}{2}\right)^2$$

$$d^{\,2}_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^{\,2}_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^{\,2}_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1)$$

$$d^{\,2}_{0,0} = \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

Figure 44.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974).

This example illustrates several features common to bound-state problems in quantum mechanics. First, we see that for any finite sphere the particle will have a positive minimum or zero-point energy. Second, we note that the particle cannot have a continuous range of energy values; the energy is restricted to discrete values corresponding to the eigenvalues of the Schrödinger equation. Third, the possible energies in this spherically symmetric problem depend on l; as is evident from the table of zeros of j_l, the minimum energy for a given l increases with l. Finally, note that the orthogonality of the j_l under the conditions of this problem shows us that the eigenfunctions corresponding to the same l but different i are orthogonal (with the weight factor corresponding to spherical polar coordinates). ■

We close this subsection with the observation that, in addition to orthogonality with respect to the scaling (to bring zeros to a specified r value), the spherical Bessel functions also possess orthogonality with respect to the indices:

$$\int_{-\infty}^{\infty} j_m(x) j_n(x) dx = 0, \quad m \neq n, \ m, \ n \geq 0. \tag{14.190}$$

The proof is left as Exercise 14.7.12. If $m = n$ (compare Exercise 14.7.13), we have

$$\int_{-\infty}^{\infty} [j_n(x)]^2 \, dx = \frac{\pi}{2n+1}. \tag{14.191}$$

The spherical Bessel functions will enter again in connection with spherical waves, but further consideration is postponed until the corresponding angular functions, the Legendre functions, have been more thoroughly discussed.

Modifed Spherical Bessel Functions

Problems involving the radial equation

$$r^2 \frac{d^2 R}{dr^2} + 2r \frac{dR}{dr} - \left[k^2 r^2 + l(l+1) \right] R = 0, \tag{14.192}$$

which differs from Eq. (14.148) only in the sign of k^2, also arise frequently in physics. The solutions to this equation are spherical Bessel functions with imaginary arguments, leading us to define **modified spherical Bessel functions** (Fig. 14.18) as follows:

$$i_n(x) = \sqrt{\frac{\pi}{2x}} I_{n+1/2}(x), \tag{14.193}$$

$$k_n(x) = \sqrt{\frac{2}{\pi x}} K_{n+1/2}(x). \tag{14.194}$$

Note that the scale factor in the definition of k_n differs from that of the other spherical Bessel functions.

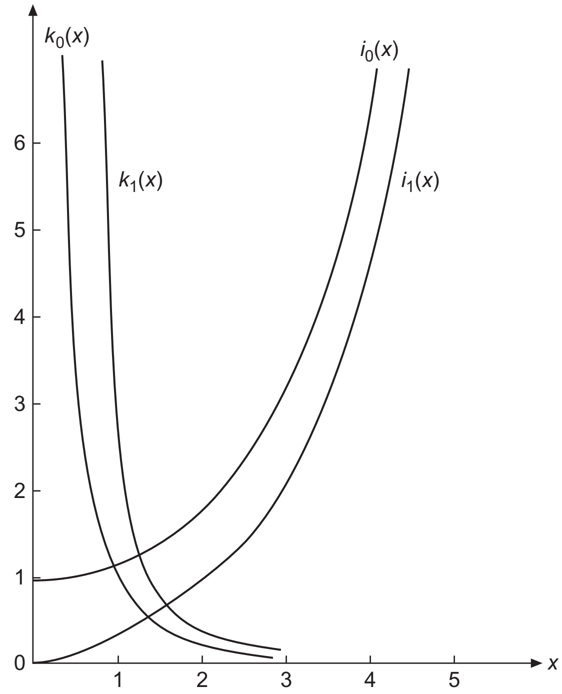


FIGURE 14.18 Modified spherical Bessel functions.

With the above definitions, these functions have the following recurrence relations:

$$\begin{aligned}
i_{n-1}(x) - i_{n+1}(x) &= \frac{2n+1}{x} i_n(x), \\
n i_{n-1}(x) + (n+1) i_{n+1}(x) &= (2n+1) i_n'(x), \\
k_{n-1}(x) - k_{n+1}(x) &= -\frac{2n+1}{x} k_n(x), \\
n k_{n-1}(x) + (n+1) k_{n+1}(x) &= -(2n+1) k_n'(x).
\end{aligned} \tag{14.195}$$

The first few of these functions are

$$\begin{aligned}
i_0(x) &= \frac{\sinh x}{x}, & k_0(x) &= \frac{e^{-x}}{x}, \\
i_1(x) &= \frac{\cosh x}{x} - \frac{\sinh x}{x^2}, & k_1(x) &= e^{-x}\left(\frac{1}{x} + \frac{1}{x^2}\right), \\
i_2(x) &= \sinh x\left(\frac{1}{x} + \frac{3}{x^3}\right) - \frac{3\cosh x}{x^2}, & k_2(x) &= e^{-x}\left(\frac{1}{x} + \frac{3}{x^2} + \frac{3}{x^3}\right).
\end{aligned} \tag{14.196}$$

Limiting values of the modified spherical Bessel functions are, for small x,

$$i_n(x) \approx \frac{x^n}{(2n+1)!!}, \quad k_n(x) \approx \frac{(2n-1)!!}{x^{n+1}}. \tag{14.197}$$

For large z, the asymptotic behavior of these functions is

$$i_n(x) \sim \frac{e^x}{2x}, \quad k_n(x) \sim \frac{e^{-x}}{x}. \tag{14.198}$$

Example 14.7.2 Particle in a Finite Spherical Well

As a final example, we return to the problem of a particle trapped in a spherical potential well of radius a (Example 14.7.1), but instead of confining the particle by a wall at potential $V = \infty$ (equivalent to requiring that its wave function ψ vanish at $r = a$), we now consider a well of finite depth, corresponding to

$$V(r) = \begin{cases} V_0 < 0, & 0 \le r \le a, \\ 0, & r > a. \end{cases}$$

If the particle can have an energy $E < 0$, it will be localized in and near the potential well, with a wave function that decays to zero as r increases to values greater than a. A simple case of this problem was one of our examples of an eigenvalue problem (Example 8.3.3), but in that case we did not proceed with enough generality to identify its solutions as Bessel functions.

This problem is governed by the Schrödinger equation, which now has the form

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V(r)\psi = E\psi.$$

This is an eigenvalue equation, to be solved for ψ and E over the full three-dimensional space, subject to the condition that ψ be continuous and differentiable for all r, and that it be normalizable (thus approaching zero asymptotically at large r). Here m is the mass of the particle and $\hbar$ is Planck's constant divided by 2π.

While this problem is more difficult than that of Example 14.7.1, it becomes manageable if we realize that it is equivalent to two separate problems for the respective regions $0 \le r \le a$ and $r > a$, within each of which the potential has a constant value, but constrained to (1) have the same eigenvalue E, and (2) connect smoothly (so the r derivative will exist) at $r = a$.

When our Schrödinger equation is processed by the method of separation of variables, we obtain as its radial component

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \left(\frac{2m}{\hbar^2}\left[E - V(r)\right] - \frac{l(l+1)}{r^2}\right)R = 0,$$

which is either the spherical Bessel equation, Eq. (14.150), or the modified spherical Bessel equation, Eq. (14.192), depending on the sign of $E - V(r)$. We see that if $V_0 < E < 0$, then