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1 Valna funkcija

1.1 U trenutku $t = 0$ kvantno stanje čestice određeno je valnom funkcijom

$$\Psi(x,0) = \begin{cases} Ax/a, & \text{za } 0 \le x \le a \\ A(b-x)/(b-a), & \text{za } a \le x \le b \\ 0, & \text{drugo} \end{cases}$$

gdje su A, a, b konstante.
(a) Normalizirajte Ψ.
(b) Nacrtajte Ψ kao funkciju od x.
(c) Gdje je najvjerojatniji položaj čestice u $t = 0$?
(d) Kolika je vjerojatnost da nađemo česticu u lijevo od $x = a$? Provjerite svoj rezultat u graničnim slučajevima $b = a$ i $b = 2a$?
(e) Kolika je prosječna vrijednost od x?

1.2 Razmotrite valnu funkciju

$$\Psi(x,t) = A\mathrm{e}^{-\lambda|x|}\mathrm{e}^{-i\omega t}$$

gdje su A, λ i ω pozitivne realne konstante.
(a) Normalizirajte Ψ.
(b) Odredite prosječne vrijednosti od x i x^2.
(c) Nađite standardnu devijaciju od x. Skicirajte graf od $|\Psi|^2$ kao funkciju od x, te označite točke $\langle x\rangle - \sigma$ i $\langle x\rangle + \sigma$ da pokažete u kojem smislu σ ukazuje na 'rasipanje' vrijednosti od x oko prosječne vrijednosti. Kolika je vjerojatnost da će čestica biti nađena izvan tog intervala?

1.3 Čestica mase m je u stanju

$$\Psi(x,t) = A\mathrm{e}^{-amx^2/\hbar}\cdot\mathrm{e}^{-iat}$$

gdje su A i a pozitivne realne konstante.
(a) Nađite A.
(b) Za koju potencijalnu energiju $V(x)$ zadana valna funkcija zadovoljava Schrödingerovu jednadžbu?
(c) Izračunajte prosječne vrijednosti od x, x^2, p i p^2.
(d) Nađite σ_x i σ_p . Je li njihov produkt u suglasju sa relacijom neodređenosti?

1.4 (a) Pokažite da Schrödingerova jednadžba čuva normalizaciju, odnosno da je

$$\int |\Psi(x,t)|^2\,\mathrm{d}x = 1$$

za svaki trenutak t.
(b) Upotrijebite Schrödingerovu jednadžbu da dokažete jednakost

$$\langle p\rangle = m\frac{\mathrm{d}\langle x\rangle}{\mathrm{d}t} = \frac{\hbar}{i}\int_{-\infty}^{\infty}\Psi^*\frac{\partial\Psi}{\partial x}\mathrm{d}x \;.$$

1.5 Neka je $P_{ab}(t)$ vjerojatnost nalaženja čestice u intervalu ($a < x < b$), u trenutku t.
(a) Pokažite da vrijedi

$$\frac{\mathrm{d}P_{ab}}{\mathrm{d}t} = J(a,t) - J(b,t)$$

gdje je

$$J(x,t) \equiv \frac{i\hbar}{2m}\left(\Psi\frac{\partial\Psi^*}{\partial x} - \Psi^*\frac{\partial\Psi}{\partial x}\right) \;.$$

Koja je jedinica za $J(x, t)$? Veličina J se naziva *strujom vjerojatnosti* zato jer pokazuje brzinu kojom vjerojatnost "protječe" u točki x. Ako $P_{ab}(t)$ raste, tada više vjerojatnosti teče u područje oko x nego što izlazi.
(b) Nađite struju vjerojatnosti za valnu funkciju

$$\Psi(x,t) = A\mathrm{e}^{-\lambda|x|}\mathrm{e}^{-i\omega t}$$

1.6 Pretpostavimo da želite opisati *nestabilnu česticu* koja se spontano raspada s "vremenom života" τ. U tom slučaju ukupna vjerojatnost za nalaženje čestice u nekom dijelu prostora ne bi smjela biti konstantna, već se treba smanjivati eksponencijalnom brzinom

$$P(t) \equiv \int_{-\infty}^{+\infty} |\Psi(x,t)|^2 \,\mathrm{d}x = \mathrm{e}^{-t/\tau}$$

"Grubi" način na koji možemo doći do gornje formule je sljedeći: uobičajeno pretpostavljamo da je potencijalna energija V *realna*. Takva pretpostavka koja je veoma logična, dovodi do *očuvanja vjerojatnosti* što je sadržaj jednadžbe

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{+\infty} |\Psi(x,t)|^2 \,\mathrm{d}x = 0 \qquad (*)$$

Tu smo tvrdnju dokazali u Problemu 1.4. Recimo da potencijalnoj energiji V dodamo i imaginarni dio $V \rightarrow V = V_0 - i\Gamma$, gdje je V_0 prava (realna) potencijalna energija, a Γ pozitivna i realna konstanta.
(a) Pokažite da umjesto (*) dobijemo

$$\frac{\mathrm{d}P}{\mathrm{d}t} = -\frac{2\Gamma}{\hbar}P$$

(b) Riješite ovu jednadžbu i nađite vrijeme života čestice izraženog pomoću Γ.

$\boxed{1.1}$

(a) Normalizacija

$$\int_0^b |\psi|^2 dx = 1$$

$$\frac{|A|^2}{a^2}\int_0^a x^2 dx + \frac{|A|^2}{(b-a)^2}\int_a^b (b-x)^2 dx = 1$$

Integrali su jednaki

$$\int_0^a x^2 dx = \frac{x^3}{3}\Big/_0^a = \frac{a^3}{3}$$

$$\int_a^b (b-x)^2 dx = [\, b-x = y \,]$$

$$= -\int_{b-a}^{0} y^2 dy = \frac{y^3}{3}\Big/_0^{b-a} = \frac{(b-a)^3}{3}$$

Imamo

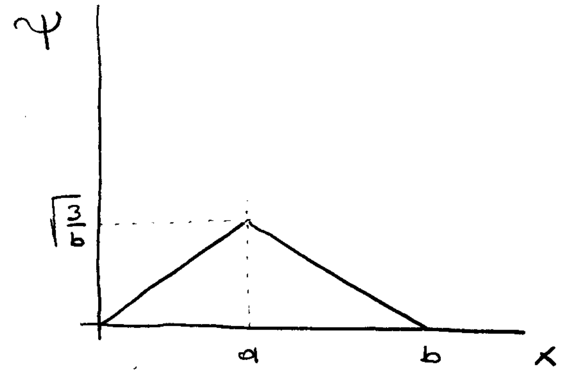
$$\frac{|A|^2}{a^2}\cdot\frac{a^3}{3} + \frac{|A|^2}{(b-a)^2}\cdot\frac{(b-a)^3}{3} = \frac{|A|^2}{3} b = 1$$

$$\Longrightarrow |A|^2 = \frac{3}{b}$$

Valna funkcija je neodređena do na fazni faktor, stoga smijemo uzeti da je A realan

$$A = \sqrt{\frac{3}{b}}$$

(b)



$$\psi = \begin{cases} \frac{1}{a}\sqrt{\frac{3}{b}}\, x ; & 0 \le x \le a \\ \frac{1}{(b-a)}\sqrt{\frac{3}{b}}\,(b-x) ; & a \le x \le b \\ 0 ; & \text{drugo} \end{cases}$$

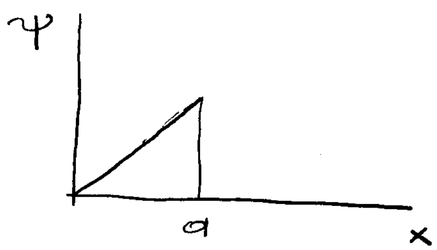
(c) Prema grafu valne funkcije, gustoća vjerojatnosti je maksimalna u $x=a$, pa je najvjerojatniji položaj čestice u $x=a$.

(d) Vjerojatnost da česticu nađemo lijevo od a jednaka je

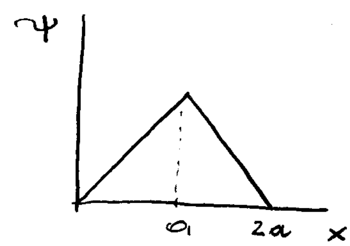
$$P_{[0,a]} = \int_0^a |\psi|^2 dx = \int_0^a \frac{1}{a^2}\cdot\frac{3}{b}\, x^2 dx$$

$$= \frac{3}{a^2 b}\cdot \frac{x^3}{3}\Big|_0^a = \frac{a}{b}$$

Za $b \to a$, $P_{[0,a]} \to 1$



Za $b \to 2a$, $P_{[0,a]} \to \frac{1}{2}$



(e) Tražimo

$$\langle x\rangle = \int_0^b x\,|\psi|^2 dx = \underbrace{\int_0^a \frac{1}{a^2}\cdot\frac{3}{b}\, x^3 dx}_{I_1} + \underbrace{\int_a^b \frac{1}{(b-a)^2}\cdot\frac{3}{b}\,(b-x)^2 x\, dx}_{I_2}$$

$$I_1 = \frac{3}{a^2 b}\cdot\frac{x^4}{4}\Big|_0^a = \frac{3}{4}\,\frac{a^2}{b}$$

$$I_2 = \frac{3}{b(b-a)^2}\int_a^b \left(b^2x - 2bx^2 + x^3\right)dx$$

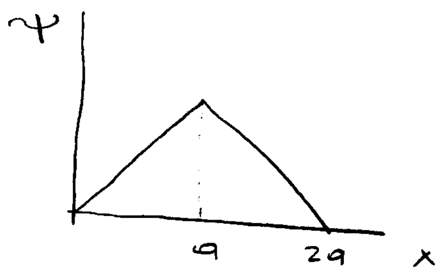
$$= \frac{3}{b(b-a)^2}\left\{ b^2 \cdot \frac{x^2}{2} - 2b\cdot\frac{x^3}{3} + \frac{x^4}{4}\right\}\Big/_a^b$$

$$= \frac{3}{b(b-a)}\left\{ \frac{b^3}{12} + \frac{3a^3}{12} + \frac{ab^2}{12} - \frac{5}{12}a^2b\right\}$$

$$= \frac{(b-a)(3a+b)}{4b}$$

$$\langle x\rangle = \frac{3}{4}\frac{a^2}{b} + \frac{(b-a)(3a+b)}{4b} = \frac{1}{4}(b+2a)$$

Za $b \to 2a$ dobivamo $\langle x \rangle = a$



$\boxed{1.2}$

(a) Normalizacija za funkciju

$$\psi(x,t) = A e^{-\lambda|x|} e^{-i\omega t}$$

Zbog apsolutne vrijednosti možemo je zapisati u obliku

$$\psi(x,t) = \begin{cases} A e^{-\lambda x} \cdot e^{-i\omega t} \,; & x \geq 0 \\ A e^{\lambda x} \cdot e^{-i\omega t} \,; & x \leq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} |\psi|^2 dx = 1$$

$$|A|^2 \int_{-\infty}^{0} \underbrace{e^{2\lambda x} dx}_{x \to -x} + |A|^2 \int_{0}^{\infty} e^{-2\lambda x} dx = 2|A|^2 \int_{0}^{\infty} e^{-2\lambda x} dx =$$

$$= 2|A|^2 \frac{e^{-2\lambda x}}{(-2\lambda)} \Big/_{0}^{\infty}$$

$$= \frac{|A|^2}{\lambda} = 1$$

$$\Rightarrow \quad |A|^2 = \lambda$$

$$A = \sqrt{\lambda}$$

(b)

$$\langle x \rangle = \int_{-\infty}^{\infty} x |\psi|^2 dx = \lambda \int_{-\infty}^{\infty} x e^{-2\lambda|x|} dx$$

Podintegralna funkcija je neparna pa je integral jednak nuli

$$\langle x \rangle = 0$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 |\psi|^2 dx = \lambda \int_{-\infty}^{0} e^{2\lambda x} x^2 dx + \lambda \int_{0}^{\infty} e^{-2\lambda x} \cdot x^2 dx$$

Oba integrala daju jednaku vrijednost. Dovoljno je računati

$$\int_0^{\infty} e^{-2\lambda x} \cdot x^2 dx = e^{-2\lambda x}\left(\frac{x^2}{2\lambda} - \frac{2x}{4\lambda^2} - \frac{2}{8\lambda^3}\right)\Big/_0^{\infty}$$

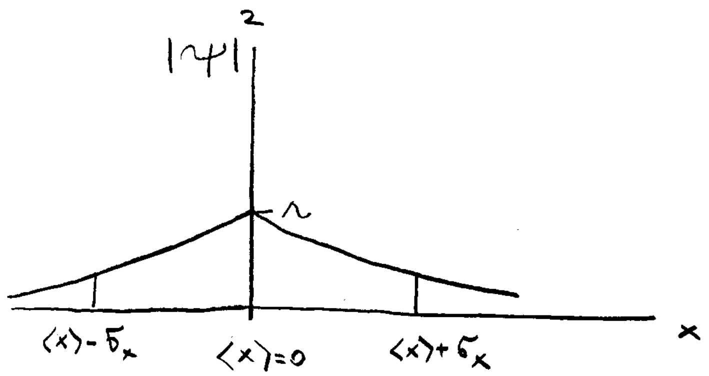
$$= \frac{1}{4\lambda^3}$$

$$\langle x^2\rangle = 2\lambda \cdot \frac{1}{4\lambda^3} = \frac{1}{2\lambda^2}$$

(c) Neodređenost položaja (standardna devijacija)

$$\sigma_x = \sqrt{\underbrace{\langle x^2\rangle}_{\frac{1}{2\lambda^2}} - \underbrace{\langle x\rangle^2}_{=0}} = \frac{1}{\lambda\sqrt{2}}$$

$$|\psi|^2 = \lambda e^{-2\lambda|x|}$$



Gustoća vjerojatnosti u točkama

$$x = \pm \sigma_x$$

$$|\psi|^2 = \lambda e^{-\sqrt{2}} = 0{,}243 \cdot \lambda$$

Gustoća vjerojatnosti je pala na četvrtinu maksimalne vrijednosti.

Vjerojatnost da česticu nađemo izvan $(\langle x\rangle - \sigma_x , \langle x\rangle + \sigma_x)$

$$\mathcal{P} = \lambda \int_{-\infty}^{-\frac{1}{\lambda\sqrt{2}}} e^{2\lambda x} dx + \lambda \int_{\frac{1}{\lambda\sqrt{2}}}^{\infty} e^{-2\lambda x} dx$$

Dovoljno je izračunati jedan od gornja dva integrala: oba su jednaka.

$$\int_{\frac{1}{\lambda\sqrt{2}}}^{\infty} e^{-2\lambda x} dx = \frac{e^{-2\lambda x}}{-2\lambda}\Big/_{\frac{1}{\lambda\sqrt{2}}}^{\infty} = \frac{1}{2\lambda} e^{-2\lambda\frac{1}{\lambda\sqrt{2}}} = \frac{1}{2\lambda} e^{-\sqrt{2}}$$

Vjerojatnost je jednaka

$$P = 2\lambda \cdot \frac{1}{2\lambda} \cdot e^{-\sqrt{2}} = 0{,}243$$

1.3

(a) Normalizacija valne funkcije

$$\psi(x,t) = A e^{-amx^2/\hbar} \cdot e^{-iat}$$

$$\int_{-\infty}^{\infty} |\psi|^2 dx = 1$$

$$|A|^2 \int_{-\infty}^{\infty} e^{-\frac{2am}{\hbar}x^2} dx = |A|^2 \cdot 2 \frac{\sqrt{\pi}}{2\sqrt{\frac{2am}{\hbar}}} = 1$$

$$A = \left(\frac{2am}{\pi \cdot \hbar}\right)^{1/4}$$

(b)

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x)\psi = i\hbar \frac{\partial \psi}{\partial t}$$

$$\frac{\partial \psi}{\partial x} = A e^{-amx^2/\hbar} \cdot e^{-iat} \cdot \left(-\frac{2am}{\hbar} x\right)$$

$$\frac{\partial^2 \psi}{\partial x^2} = A e^{-iat} \cdot e^{-amx^2/\hbar} \cdot \frac{4a^2m^2}{\hbar^2} x^2 + A e^{-iat} \cdot e^{-amx^2/\hbar} \left(-\frac{2am}{\hbar}\right)$$

$$\frac{\partial \psi}{\partial t} = A e^{-amx^2/\hbar} \cdot e^{-iat} \cdot (-ia)$$

Uvrstimo u Schrödingerovu jednadžbu

$$-\frac{\hbar^2}{2m} \left\{ \frac{4a^2m^2}{\hbar^2} x^2 - \frac{2am}{\hbar} \right\} \psi + V(x)\psi = a\hbar \psi$$

$$\Rightarrow -2a^2mx^2 + V(x) = 0$$

$$V(x) = 2a^2mx^2$$

(c)

$$\langle x \rangle = |A|^2 \int_{-\infty}^{\infty} e^{-\frac{2am}{\hbar}x^2} x \, dx$$

Podintegralna funkcija je neparna: $\langle x \rangle = 0$

$$\langle x^2\rangle = |A|^2 \int_{-\infty}^{\infty} e^{-\frac{2am}{\hbar}x^2} x^2 dx = 2|A|^2 \frac{\Gamma(\frac{3}{2})}{2\left(\frac{2am}{\hbar}\right)^{3/2}}$$

$$= \left(\frac{2am}{\pi\hbar}\right)^{1/2} \cdot \frac{\frac{1}{2}\sqrt{\pi}}{\left(\frac{2am}{\hbar}\right)^{3/2}} = \frac{\hbar}{4am}$$

$\langle p\rangle = m\frac{d}{dt}\langle x\rangle = 0$ jer se srednja vrijednost uzima u stacionarnom stanju.

$$\langle p^2\rangle = \int_{-\infty}^{\infty} \psi^*\left(-\hbar^2\frac{\partial^2\psi}{\partial x^2}\right)dx$$

$$= -\hbar^2|A|^2\left\{\int_{-\infty}^{\infty} e^{-\frac{2am}{\hbar}x^2}\cdot\frac{4a^2m^2}{\hbar^2}x^2dx + \int_{-\infty}^{\infty} e^{-\frac{2am}{\hbar}x^2}\cdot\left(-\frac{2am}{\hbar}\right)dx\right\}$$

$$= -\hbar^2\cdot\left\{\frac{4a^2m^2}{\hbar^2}\cdot\frac{\hbar}{4am} - \frac{2am}{\hbar}\right\} = am\hbar$$

$$\sigma_x^2 = \langle x^2\rangle - \langle x\rangle^2 = \frac{\hbar}{4am} \Rightarrow \sigma_x = \sqrt{\frac{\hbar}{4am}}$$

$$\sigma_p^2 = \langle p^2\rangle - \langle p\rangle^2 = am\hbar \Rightarrow \sigma_p = \sqrt{am\hbar}$$

$\sigma_x\sigma_p = \frac{\hbar}{2}$ Princip neodređenosti je zadovoljen.

$\boxed{1.4}$

(a) Treba pokazati da je

$$\frac{d}{dt}\int_{-\infty}^{\infty}|\psi|^2 dx = \int_{-\infty}^{\infty}\frac{\partial}{\partial t}|\psi|^2 dx = 0$$

$$\frac{\partial}{\partial t}|\psi|^2 = \frac{\partial}{\partial t}\psi^*\psi = \frac{\partial\psi^*}{\partial t}\psi + \psi^*\frac{\partial\psi}{\partial t}$$

Vremensku derivaciju eliminirat ćemo iz Schrödingerove jednadžbe

$$\frac{\partial\psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2\psi}{\partial x^2} - \frac{i}{\hbar}V\psi$$

$$\frac{\partial\psi^*}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2\psi^*}{\partial x^2} + \frac{i}{\hbar}V\psi^*$$

pri čemu smo pretpostavili da je potencijalna energija <u>realna</u>

$$V = V^*$$

$$\frac{\partial}{\partial t}|\psi|^2 = -\frac{i\hbar}{2m}\frac{\partial^2\psi^*}{\partial x^2}\psi + \frac{i}{\hbar}V\psi^*\psi + \frac{i\hbar}{2m}\frac{\partial^2\psi}{\partial x^2}\psi^* - \frac{i}{\hbar}V\psi\psi^*$$

$$= \frac{i\hbar}{2m}\left(\frac{\partial^2\psi}{\partial x^2}\psi^* - \frac{\partial^2\psi^*}{\partial x^2}\psi\right)$$

$$= \frac{i\hbar}{2m}\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial x}\psi^* - \frac{\partial\psi^*}{\partial x}\psi\right)$$

$$\frac{d}{dt}\int_{-\infty}^{\infty}|\psi|^2 dx = \frac{i\hbar}{2m}\int_{-\infty}^{\infty}\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial x}\psi^* - \frac{\partial\psi^*}{\partial x}\psi\right)dx$$

$$= \frac{i\hbar}{2m}\left(\frac{\partial\psi}{\partial x}\psi^* - \frac{\partial\psi^*}{\partial x}\psi\right)\Big|_{-\infty}^{\infty}$$

Valna funkcija mora biti <u>normalizabilna</u>

$$\psi, \psi^* \xrightarrow{x\to\pm\infty} 0$$

u protivnom bi integral $\int_{-\infty}^{\infty}|\psi|^2 dx$ divergirao. Uz navedeni uvjet imamo

$$\frac{d}{dt}\int_{-\infty}^{\infty}|\psi|^2 dx = \frac{i\hbar}{2m}\left(\frac{\partial\psi}{\partial x}\psi^* - \frac{\partial\psi^*}{\partial x}\psi\right)\Big|_{-\infty}^{\infty} \to 0$$

pa je tvrdnja dokazana.

NAPOMENA: upravo dokazanu jednakost nismo dokazali diferenciranjem jednakosti

$$\int_{-\infty}^{+\infty} |\psi|^2 dx = 1$$

već sasvim suprotno: gornja jednakost je moguća samo onda ako je

$$\frac{d}{dt}\int_{-\infty}^{+\infty} |\psi|^2 dx = 0$$

(b) Definicija prosječne vrijednosti impulsa

$$\langle p\rangle \equiv m\frac{d\langle x\rangle}{dt} = m\frac{d}{dt}\int_{-\infty}^{+\infty} x|\psi|^2 dx$$

$$= m\int_{-\infty}^{+\infty} x\frac{\partial}{\partial t}|\psi|^2 dx$$

Iz zadatka 1.1(a) slijedi

$$\frac{\partial}{\partial t}|\psi|^2 = \frac{i\hbar}{2m}\frac{\partial}{\partial x}\underbrace{\left(\frac{\partial\psi}{\partial x}\psi^* - \frac{\partial\psi^*}{\partial x}\psi\right)}_{f}$$

Pod integralom pavlja se

$$x\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(xf) - \frac{\partial x}{\partial x}f = \frac{\partial}{\partial x}(xf) - f$$

Stoga, parcijalna integracija daje

$$\int_{-\infty}^{+\infty} x\frac{\partial f}{\partial x}dx = xf\Big/_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} f dx$$

No,

$$xf\Big/_{-\infty}^{\infty} = x\frac{\partial\psi}{\partial x}\psi^* - x\frac{\partial\psi^*}{\partial x}\psi \to 0$$

zbog uvjeta normalizabilnosti valne funkcije ψ. Ostaje

$$\langle p\rangle = -\frac{i\hbar}{2}\int_{-\infty}^{+\infty}\left(\frac{\partial\psi}{\partial x}\psi^* - \frac{\partial\psi^*}{\partial x}\psi\right)dx$$

Podintegralnu funkciju možemo napisati u obliku

$$\frac{\partial}{\partial x}(\psi\psi^*) = \frac{\partial\psi}{\partial x}\psi^* + \psi\frac{\partial\psi^*}{\partial x}$$

$$\int_{-\infty}^{\infty}\left(\frac{\partial\psi}{\partial x}\psi^* - \frac{\partial\psi^*}{\partial x}\psi\right)dx = -\int_{-\infty}^{\infty}\frac{\partial}{\partial x}(\psi\psi^*)dx + 2\int_{-\infty}^{\infty}\psi^*\frac{\partial\psi}{\partial x}dx$$

Integral

$$\int_{-\infty}^{\infty}\frac{\partial}{\partial x}(\psi\psi^*)\,dx = \psi\psi^*\Big/_{-\infty}^{\infty}$$

jednak je nuli zbog $\psi \xrightarrow{\pm\infty} 0$. Dobijamo

$$\langle p\rangle = -i\hbar\int_{-\infty}^{\infty}\psi^*\frac{\partial\psi}{\partial x}dx = \int_{-\infty}^{\infty}\psi^*\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\psi\,dx$$

$\boxed{1.5}$

(a) $$P_{ab} = \int_a^b |\psi|^2 dx$$

$$\frac{dP_{ab}}{dt} = \int_a^b \frac{\partial}{\partial t}|\psi|^2 dx$$

$$= [\text{zadatak } 1.1(a)]$$

$$= -\frac{i\hbar}{2m}\left(\psi \frac{\partial \psi^*}{\partial x} - \frac{\partial \psi}{\partial x}\psi^*\right)\Big|_a^b$$

Definicija struje vjerojatnosti

$$J(x,t) \equiv \frac{i\hbar}{2m}\left(\psi \frac{\partial \psi^*}{\partial x} - \psi^* \frac{\partial \psi}{\partial x}\right).$$

$$\frac{dP_{ab}}{dt} = -J(b,t) + J(a,t)$$

Jedinica za struju vjerojatnosti je s^{-1}, što se vidi iz lijeve strane gornje jednakosti.

(b) $$\psi(x,t) = A e^{-\lambda|x|} e^{-i\omega t} = A e^{-i\omega t} \cdot \begin{cases} e^{-\lambda x}; & x \geq 0 \\ e^{\lambda x}; & x \leq 0 \end{cases}$$

$x \geq 0;$ $$\frac{\partial \psi_+}{\partial x} = A \cdot (-\lambda) e^{-\lambda x} e^{-i\omega t}$$

$x \leq 0;$ $$\frac{\partial \psi_-}{\partial x} = A \cdot \lambda e^{\lambda x} e^{-i\omega t}$$

Struja vjerojatnosti za $x \geq 0$

$$J_+(x,t) = \frac{i\hbar}{2m}\left(A e^{-\lambda x} e^{-i\omega t} \cdot A^*(-\lambda) \cdot e^{-\lambda x} e^{i\omega t} - A^* e^{-\lambda x} e^{i\omega t} \cdot A \cdot (-\lambda) e^{-\lambda x} e^{-i\omega t}\right)$$

$$= 0$$

Slično, $J_- = 0$; dakle $J = 0$ na cijelom intervalu.

1.6

(a) $$P(t) \equiv \int_{-\infty}^{\infty} |\psi|^2 dx$$

$$\frac{dP}{dt} = \frac{d}{dt}\int_{-\infty}^{\infty} |\psi|^2 dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial t}|\psi|^2 dx$$

Slijedimo račun iz zadatka 1.4 (a)

$$\frac{\partial}{\partial t}|\psi|^2 = \frac{\partial}{\partial t}(\psi^*\psi) = \frac{\partial \psi^*}{\partial t}\psi + \psi^* \frac{\partial \psi}{\partial t}$$

Schrödingerova jednadžba

$$\frac{\partial \psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 \psi}{\partial x^2} - \frac{i}{\hbar} V\psi$$

$$\frac{\partial \psi^*}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2 \psi^*}{\partial x^2} + \frac{i}{\hbar} V^*\psi^*$$

$$\frac{\partial}{\partial t}|\psi|^2 = \frac{i\hbar}{2m}\left(\frac{\partial^2 \psi}{\partial x^2}\psi^* - \frac{\partial^2 \psi^*}{\partial x^2}\psi\right) + \frac{i}{\hbar}\psi\psi^*(V^* - V)$$

Zbog $V = V_0 - i\Gamma$ imamo

$$V^* - V = 2i\Gamma$$

$$\frac{\partial}{\partial t}|\psi|^2 = \frac{i\hbar}{2m}\left(\frac{\partial^2 \psi}{\partial x^2}\psi^* - \frac{\partial^2 \psi^*}{\partial x^2}\psi\right) - \frac{2\Gamma}{\hbar}|\psi|^2$$

Iz zadatka 1.4 (a) znamo da je

$$\int_{-\infty}^{\infty}\left(\frac{\partial^2 \psi}{\partial x^2}\psi^* - \frac{\partial^2 \psi^*}{\partial x^2}\psi\right) dx \to 0$$

Ostaje,

$$\frac{dP}{dt} = \int_{-\infty}^{\infty} \frac{\partial}{\partial t}|\psi|^2 dx = -\frac{2\Gamma}{\hbar}\underbrace{\int_{-\infty}^{\infty}|\psi|^2 dx}_{P(t)} = -\frac{2\Gamma}{\hbar}P(t)$$

Dobijamo jednadžbu

$$\frac{dP}{dt} = -\frac{2\Gamma}{\hbar}P(t)$$

(b) Rješenje ove jednadžbe je

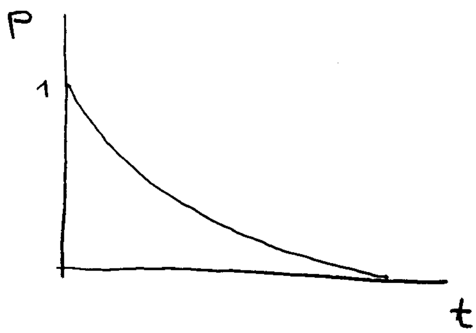
$$P(t) = A\, e^{-\frac{2\Gamma}{\hbar} t}$$

Uzmimo: neka je $P(t=0)=1$. Imamo

$$P(0) = A = 1$$

Time smo odredili konstantu A. Za vrijeme života čestice dobivamo

$$\tau = \frac{\hbar}{2\Gamma}$$



Vjerojatnost da nađemo česticu u trenutku t blizo tme.