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7 Ket i bra vektori. Hermitski operatori

7.1 Vektori $|\alpha\rangle$ i $|\beta\rangle$ iz istog prostora zapisani su u različitim bazama $\{|e_i\rangle\}$, $\{|f_i\rangle\}$

$$|\alpha\rangle = \sum_i |e_i\rangle\langle e_i|\alpha\rangle$$

$$|\beta\rangle = \sum_i |f_i\rangle\langle f_i|\beta\rangle$$

(a) Napišite ket $|\beta\rangle$ u bazi $\{|e_i\rangle\}$ ako vrijedi $\langle e_i|f_j\rangle = M\delta_{ij}$, gdje je M realan broj.
(b) Nađite skalarni produkt $\langle\alpha|\beta\rangle$. Koliki bi rezultat dobili da smo računali skalarni produkt u drugoj bazi?

7.2 Operator projekcije spina 1/2 na x-os S_x napisan je u bazi $\{|\uparrow\rangle, |\downarrow\rangle\}$

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Napomena**:** vektori $\{|\uparrow\rangle, |\downarrow\rangle\}$ su svojstveni vektori operatora projekcije spina na z-os, S_z .
(a) Napišite djelovanje operatora S_x na vektore u bazi.
(b) Ako je

$$|\alpha\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle$$

$$|\beta\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\rangle$$

napišite koliko je $S_x|\alpha\rangle$, $S_x|\beta\rangle$.
(c) Da li vektori $|\alpha\rangle$, $|\beta\rangle$ čine bazu?
(d) Zapišite operator S_x u bazi $\{|\alpha\rangle, |\beta\rangle\}$.

7.3 Zadan je operator A u bazi $\{|\eta\rangle\}$. Napišite jednadžbu iz koje se određuju svojstvene vrijednosti za A. Kakva je to jednadžba?

7.4 Raspišite lijevu stranu relacije potpunosti

$$\sum_{i=1}^{3} |e_i\rangle\langle e_i| = I$$

u ortonormiranoj bazi $\{|e_i\rangle\}$ i uvjerite se da zaista vrijedi.

7.5 (a) Zadan je hermitski operator A. Pokažite da je $\langle\psi|A^2|\psi\rangle \geq 0$ za po volji uzeti vektor stanja $|\psi\rangle$ iz prostora na kojem djeluje A. Pomoću dokazanog teorema odgovorite na pitanje je li prosječna vrijednost kinetičke energije uvijek veća ili jednaka nuli.
(b) Pokažite da je prosječna vrijednost opservable, fizikalne veličine opisane hermitskim operatorom, uvijek realna.

7.6 (a) Jesu li operatori podizanja i spuštanja definirani kod harmoničkog oscilatora hermitski operatori?
(b) Iz definicija operatora podizanja i spuštanja zaključujemo da je

$$(a_+)^\dagger = a_-$$

pa možemo definirati novi operator

$$a_- \equiv a$$

$$a_+ \equiv a^\dagger$$

Pronađite svojstvene funkcije i svojstvene vrijednosti operatora a. Kvantna stanja koja su opisana svojstvenim funkcijama od a nazivaju se *koherentnim stanjima*.

$\boxed{7.1}$

(a) Želimo napisati ket $|z\rangle$ u bazi $\{|e_i\rangle\}$. Općenito,

$$|z\rangle = \sum_i c_i |e_i\rangle$$

gdje su c_i koeficijenti koje treba odrediti. Pomnožimo cijelu jednadžbu slijeva sa $\langle f_j|$

$$\langle f_j|z\rangle = \sum_i c_i \langle f_j|e_i\rangle = \sum_i c_i M \delta_{ij} = M c_j$$

Odavdje,

$$c_j = \frac{1}{M} \overbrace{\langle f_j|z\rangle}^{\text{poznato}}$$

Zapis vektora $|z\rangle$ u bazi $\{|e_i\rangle\}$ glasi

$$|z\rangle = \frac{1}{M} \sum_i |e_i\rangle\langle f_i|z\rangle$$

(b) Ket vektoru

$$|\alpha\rangle = \sum_i |e_i\rangle\langle e_i|\alpha\rangle$$

odgovara bra

$$\langle\alpha| = \sum_i \langle e_i|\alpha\rangle^* \langle e_i| = \sum_i \langle\alpha|e_i\rangle\langle e_i|$$

Skalarni produkt

$$\langle\alpha|z\rangle = \left(\sum_i \langle\alpha|e_i\rangle\langle e_i|\right)\cdot\left(\sum_j |f_j\rangle\langle f_j|z\rangle\right)$$

$$= \sum_i \sum_j \langle\alpha|e_i\rangle\underbrace{\langle e_i|f_j\rangle}_{M\delta_{ij}}\langle f_j|z\rangle$$

$$= M \sum_i \langle\alpha|e_i\rangle\langle f_i|z\rangle$$

$\boxed{7.2}$

(a) $S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$S_x|\uparrow\rangle = ?\ ;\ S_x|\downarrow\rangle = ?$

$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\ ;\ |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$S_x|\uparrow\rangle = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\hbar}{2}|\downarrow\rangle$$

$$S_x|\downarrow\rangle = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}|\uparrow\rangle$$

(b)

$$S_x|\alpha\rangle = S_x\left(\frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle\right) = \frac{1}{\sqrt{2}} S_x|\uparrow\rangle + \frac{1}{\sqrt{2}} S_x|\downarrow\rangle$$
$$= \frac{1}{\sqrt{2}}\cdot\frac{\hbar}{2}|\downarrow\rangle + \frac{1}{\sqrt{2}}\cdot\frac{\hbar}{2}|\uparrow\rangle = \frac{\hbar}{2}\left(\frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle\right)$$
$$= \frac{\hbar}{2}|\alpha\rangle$$

$$S_x|\beta\rangle = S_x\left(\frac{1}{\sqrt{2}}|\uparrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\rangle\right) = \frac{1}{\sqrt{2}} S_x|\uparrow\rangle - \frac{1}{\sqrt{2}} S_x|\downarrow\rangle$$
$$= \frac{1}{\sqrt{2}}\cdot\frac{\hbar}{2}|\downarrow\rangle - \frac{1}{\sqrt{2}}\frac{\hbar}{2}|\uparrow\rangle = \left(-\frac{\hbar}{2}\right)\left(\frac{1}{\sqrt{2}}|\uparrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\rangle\right)$$
$$= \left(-\frac{\hbar}{2}\right)|\beta\rangle$$

(c) Čine ortonormiranu bazu.

$$\langle\uparrow|\downarrow\rangle = (1,0)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

$$\langle\uparrow|\uparrow\rangle = (1,0)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$$

$$\langle\alpha|\beta\rangle = \left(\frac{1}{\sqrt{2}}\right)^2\cdot\left(\langle\uparrow| + \langle\downarrow|\right)\left(|\uparrow\rangle - |\downarrow\rangle\right)$$
$$= \frac{1}{2}\cdot\left(1 - 0 + 0 - 1\right) = 0$$ ortogonalni su.

(d) Treba pronaći matrične elemente

$$\langle \alpha | S_x | \alpha \rangle = (S_x)_{11}$$

$$= \left(\frac{1}{\sqrt{2}}\right)^2 \left(\langle \uparrow | + \langle \downarrow |\right) S_x \left(|\uparrow\rangle + |\downarrow\rangle\right)$$

$$= \frac{1}{2} \left(\langle \uparrow | + \langle \downarrow |\right)\left(\frac{\hbar}{2}|\downarrow\rangle + \frac{\hbar}{2}|\uparrow\rangle\right)$$

$$= \frac{\hbar}{4}\left(0 + 1 + 1 + 0\right) = \frac{\hbar}{2}$$

$$\langle \alpha | S_x | \beta \rangle = (S_x)_{12}$$

$$= \left(\frac{1}{\sqrt{2}}\right)^2 \left(\langle \uparrow | + \langle \downarrow |\right) S_x \left(|\uparrow\rangle - |\downarrow\rangle\right)$$

$$= \frac{1}{2} \left(\langle \uparrow | + \langle \downarrow |\right)\left(\frac{\hbar}{2}|\downarrow\rangle - \frac{\hbar}{2}|\uparrow\rangle\right)$$

$$= \frac{\hbar}{4}\left(0 - 1 + 1 + 0\right) = 0$$

$$\langle \beta | S_x | \alpha \rangle = 0$$

$$\langle \beta | S_x | \beta \rangle = -\frac{\hbar}{2}$$

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\boxed{7.3}$

$$A|\eta\rangle = \int A(\eta',\eta)|\eta'\rangle d\eta'$$

$$A|\alpha\rangle = \lambda|\alpha\rangle \qquad (*)$$

$$|\alpha\rangle = \int d\eta\, |\eta\rangle\langle\eta|\alpha\rangle$$

$$A|\alpha\rangle = \int d\eta\, A|\eta\rangle\langle\eta|\alpha\rangle = \int d\eta \int d\eta'\, A(\eta',\eta)\, |\eta'\rangle\langle\eta|\alpha\rangle$$

$$\lambda|\alpha\rangle = \lambda \int d\eta' |\eta'\rangle\langle\eta'|\alpha\rangle$$

Uvrstimo u jednadžbu vlastitih vrijednosti (*)

$$\int d\eta' \left\{ \int A(\eta',\eta)\langle\eta|\alpha\rangle\, d\eta - \lambda\langle\eta'|\alpha\rangle \right\} |\eta'\rangle = 0$$

Dobivamo jednadžbu

$$\int A(\eta',\eta)\langle\eta|\alpha\rangle d\eta = \lambda\langle\eta'|\alpha\rangle$$

Pomoću valnih funkcija $\psi(\eta) = \langle\eta|\alpha\rangle$

$$\int A(\eta',\eta)\psi(\eta)\, d\eta = \lambda\psi(\eta')$$

Dobivamo, općenito, integralnu jednadžbu.

$\boxed{7.4}$

$$|e_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \quad |e_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \quad |e_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$|e_1\rangle\langle e_1| = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (1,0,0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$|e_2\rangle\langle e_2| = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$|e_3\rangle\langle e_3| = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\sum_{i=1}^{3} |e_i\rangle\langle e_i| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

$\boxed{7.5}$

$A = A^{+}$

(a) $\langle\psi|A^2|\psi\rangle = \langle\psi|A\,A|\psi\rangle = (\underbrace{\langle\psi|A^{+}}_{\langle z|})(\underbrace{A|\psi\rangle}_{|z\rangle}) = \langle z|z\rangle \geq 0$

svojstvo skalarnog produkta

(b) $\langle\psi|A|\psi\rangle = \langle\psi|(A|\psi\rangle) = \{(\langle\psi|A^{+})|\psi\rangle\}^{*}$

svojstvo skalarnog produkta

$= \{\langle\psi|A^{+}|\psi\rangle\}^{*} = \langle\psi|A|\psi\rangle^{*}$

$\boxed{7.6}$

Operator opuštanja definiran je relacijom

$$\hat{a}\psi(x) \leftrightarrow \langle x | \hat{a} | \alpha \rangle = \frac{1}{\sqrt{2m}}\left(i\frac{\hbar}{i}\frac{d\psi}{dx} + m\omega x\psi\right)$$

gdje je $\psi(x) = \langle x | \alpha \rangle$. Svojstvene vrijednosti i funkcije tražimo iz

$$\hat{a}\psi(x) = \lambda\psi(x)$$

odnosno

$$\frac{d\psi}{dx} + \frac{m\omega}{\hbar}x\psi = \frac{\lambda\sqrt{2m}}{\hbar}\psi$$

Separacija varijabli

$$\frac{d\psi}{\psi} = -\frac{m\omega}{\hbar}x + \frac{\lambda\sqrt{2m}}{\hbar}$$

Integracija

$$\ln\psi = -\frac{m\omega}{2\hbar}x^2 + \frac{\lambda\sqrt{2m}}{\hbar}x + C_1$$

$$\psi = C_2 \exp\left(-\frac{m\omega}{2\hbar}x^2 + \frac{\lambda\sqrt{2m}}{\hbar}x\right)$$

Nije nužno da λ bude realan jer $\hat{a}$ nije hermitski operator.

Neka je

$$\lambda = A + iB, \quad A, B \in \mathbb{R}$$

Imamo

$$\psi = C_2 \exp\left(-\frac{m\omega}{2\hbar}x^2 + \frac{A\sqrt{2m}}{\hbar}x\right)\exp\left(\frac{iB\sqrt{2m}}{\hbar}x\right)$$

Normalizacija

$$\int_{-\infty}^{\infty} |\psi|^2 dx = 1$$

$$|C_2|^2 \int_{-\infty}^{\infty} \exp\left(-\frac{m\omega}{\hbar}x^2 + \frac{2A\sqrt{2m}}{\hbar}x\right)dx = 1$$

$$|c_2|^2 \int_{-\infty}^{\infty} \exp\left[-\frac{m\omega}{\hbar}\left(x^2 - \frac{2A\sqrt{2}}{\sqrt{m}\,\omega}x\right)\right] dx =$$

$$= |c_2|^2 \int_{-\infty}^{\infty} \exp\left[-\frac{m\omega}{\hbar}\left(x - \frac{A\sqrt{2}}{\omega\sqrt{m}}\right)^2 + \frac{2A^2}{\omega\hbar}\right] dx$$

Nova varijabla: $\eta = x - \frac{A\sqrt{2}}{\omega\sqrt{m}}$

$$= |c_2|^2 \exp\left(\frac{2A^2}{\omega\hbar}\right) \cdot \int_{-\infty}^{\infty} \exp\left[-\frac{m\omega}{\hbar}\eta^2\right] dx$$

$$= |c_2|^2 \exp\left(\frac{2A^2}{\omega\hbar}\right) \cdot \sqrt{\frac{\hbar\pi}{m\omega}} = 1$$

$$c_2 = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} \exp\left(\frac{-A^2}{\omega\hbar}\right)$$

Valna funkcija

$$\psi(x) = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} \exp\left[-\frac{m\omega}{2\hbar}\left(x - \frac{A\sqrt{2}}{\omega\sqrt{m}}\right)^2\right] \exp\left(i\,\frac{B\sqrt{2m}}{\hbar}x\right)$$

Svojstvena vrijednost operatora spuštanja je 0 kao i osnovno stanje za harmonički oscilator jer je

$$a|0\rangle = 0$$

Zaista, za $n = 0$ imamo

$$\psi = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right)$$

što je valna funkcija za osnovno stanje HO.

Svojstvene vrijednosti n mogu poprimiti bilo koju vrijednost iz kompleksne ravnine.