MATEMATIČKE METODE FIZIKE I

Prvi kolokvij 5. 12. 2023.

ZADATAK 1 Ako funkcija $f:\mathbb{R}^n \to \mathbb{R}$ zadovoljava jednadžbu

$$\sum_{i=1}^{n} x_i^2 \frac{\partial^2 f}{\partial x_i^2} + \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = 0$$

pokažite da funkcija h

$$h(x_1, x_2, ..., x_n) = f\left(e^{x_1}, e^{x_2}, ..., e^{x_n}\right)$$

zadovoljava

$$\sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2} = 0$$

ZADATAK 2 Primjenom Taylorove formule do članova drugog reda izračunajte približno izraz

$$(1,03)^{3,001}$$

***Uputa*:** promatrajte funkciju $f(x, y) = x^y$ oko točke (1, 3).

ZADATAK 3 Nađite točke maksimuma i minimuma funkcije

$$f(x, y, z) = x + y + z$$

u području $x^2 + y^2 + z^2 \le 1$. Postoje li ekstremi na otvorenom skupu $x^2 + y^2 + z^2 < 1$?

ZADATAK 4 Nađite volumen dijela prostora omeđenog sljedećim plohama:

$$z = x^2 + y^2$$

$$z = 2x^2 + 2y^2$$

$$y = x$$

$$y = x^2$$

ZADATAK 5 Diferenciranjem po parametru izračunajte integral

$$I = \int_0^{\pi/2} \ln\left(a^2 - \sin^2\theta\right) d\theta, \quad a > 1$$

$\boxed{1.}$

Funkcija $h(x_1, x_2, \dots, x_n) = f(e^{x_1}, e^{x_2}, \dots, e^{x_n})$ je kompozicija dviju funkcija

$$h(\vec{x}) = (f \circ g)(\vec{x}) \quad , \quad \vec{x} = (x_1, x_2 \dots x_n)$$

gdje je

$$(\eta_1, \eta_2, \dots, \eta_n) \to f(\eta_1, \eta_2, \dots, \eta_n)$$

$$(x_1, x_2, \dots, x_n) \to g(x_1, x_2, \dots, x_n) = (e^{x_1}, e^{x_2}, \dots, e^{x_n})$$

$$\frac{\partial h}{\partial x_i} = \frac{\partial f}{\partial \eta_i} \cdot \frac{\partial \eta_i}{\partial x_i} = \frac{\partial f}{\partial \eta_i} \cdot e^{x_i} \quad , \quad \eta_i = e^{x_i}$$

$$\frac{\partial^2 h}{\partial x_i^2} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial \eta_i} e^{x_i}\right) = \frac{\partial^2 f}{\partial \eta_i^2} \cdot \underbrace{\frac{\partial \eta_i}{\partial x_i}}_{e^{x_i}} \cdot e^{x_i} + \frac{\partial f}{\partial \eta_i} \cdot \frac{\partial}{\partial x_i} e^{x_i}$$

$$= \frac{\partial^2 f}{\partial \eta_i^2} \cdot (e^{x_i})^2 + \frac{\partial f}{\partial \eta_i} \cdot e^{x_i}$$

$$= \frac{\partial^2 f}{\partial \eta_i^2} \cdot \eta_i^2 + \frac{\partial f}{\partial \eta_i} \cdot \eta_i$$

$$\sum_i \frac{\partial^2 h}{\partial x_i^2} = \sum_i \left(\frac{\partial f}{\partial \eta_i} \eta_i^2 + \frac{\partial f}{\partial \eta_i} \eta_i\right) = 0$$

$\boxed{2.}$

Taylorov red oko točke $(1,3)$ do članova drugog reda za funkciju $f(x,y) = x^y$

$$f(x,y) = f(x_0,y_0) + \frac{1}{1!}\left[(x-x_0)\frac{\partial f}{\partial x}\Big|_{(x_0,y_0)} + \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)}(y-y_0)\right]$$

$$+ \frac{1}{2!}\left[(x-x_0)^2\frac{\partial^2 f}{\partial x^2}\Big|_{(x_0,y_0)} + 2(x-x_0)(y-y_0)\frac{\partial^2 f}{\partial x \partial y}\Big|_{(x_0,y_0)}\right.$$

$$\left. + (y-y_0)^2\frac{\partial^2 f}{\partial y^2}\Big|_{(x_0,y_0)}\right]$$

gdje je $(x_0,y_0) = (1,3)$.

Računamo derivacije:

$$\frac{\partial f}{\partial x} = y x^{y-1}; \quad \frac{\partial f}{\partial x}\Big|_{(x_0,y_0)} = 3 \cdot 1^2 = 3$$

$$\frac{\partial f}{\partial y} = x^y \ln x; \quad \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)} = 1^3 \cdot \ln 1 = 0$$

$$\frac{\partial^2 f}{\partial x^2} = y\cdot(y-1)\, x^{y-2}; \quad \frac{\partial^2 f}{\partial x^2}\Big|_{(x_0,y_0)} = 3\cdot 2 \cdot 1^1 = 6$$

$$\frac{\partial^2 f}{\partial y^2} = x^y \cdot \ln^2 x; \quad \frac{\partial^2 f}{\partial y^2}\Big|_{(x_0,y_0)} = 1^3 \ln^2 1 = 0$$

$$\frac{\partial^2 f}{\partial x \partial y} = x^y \cdot \ln^2 x + x^y \cdot \frac{1}{x}; \quad \frac{\partial^2 f}{\partial y^2}\Big|_{(x_0,y_0)} = 1^3 \cdot \ln^2 1 + 1^{3-1} = 1$$

Prema tome,

$$f(x,y) = 1 + (x-1)\cdot 3 + \frac{1}{2}\left[(x-1)^2 \cdot 6 + 2(x-1)(y-3)\cdot 1\right]$$

$$f(1{,}03;3{,}001) = 1 + (1{,}03-1)\cdot 3 + \frac{1}{2}\Big[(1{,}03-1)^2\cdot 6 + 2(1{,}03-1)(3{,}001-3)\Big]$$

$$= 1 + 3\cdot 0{,}03 + \frac{1}{2}\Big[\,0{,}03^2\cdot 6 + 2\cdot 0{,}03\cdot 0{,}001\Big]$$

$$= 1 + 0{,}09 + \frac{1}{2}\Big[\,0{,}00546\Big] = \underline{\underline{1{,}09273}}$$

$\boxed{3.}$

Funkcija f nema stacionarnih točaka jer je

$$\vec{\nabla} f \neq 0$$

u proizvoljnoj točki. To znači da f nema stacionarnih točaka na otvorenom skupu

$$x^2+y^2+z^2<1$$

Očito, min i max se javljaju na rubu tog područja tj. na sferi

$$x^2+y^2+z^2=1$$

Rješavamo metodom Lagrangeovih množitelja za funkciju

$$g = f + \lambda(1-x^2-y^2-z^2) = x+y+z + \lambda(1-x^2-y^2-z^2)$$

Imamo,

$$\frac{\partial g}{\partial x} = 1 - 2\lambda x = 0 \Rightarrow x = \frac{1}{2\lambda}$$

$$\frac{\partial g}{\partial y} = 1-2\lambda y = 0 \Rightarrow y = \frac{1}{2\lambda}$$

$$\frac{\partial g}{\partial z} = 1 - 2\lambda z = 0 \Rightarrow z = \frac{1}{2\lambda}$$

$$\frac{\partial g}{\partial \lambda} = -x^2 - y^2 - z^2 + 1 = 0$$

$$\frac{1}{(2\lambda)^2} + \frac{1}{(2\lambda)^2} + \frac{1}{(2\lambda)^2} = 1$$

$$3 = 4\lambda^2 \Rightarrow \lambda = \pm\frac{\sqrt{3}}{2}$$

$$\boxed{\lambda = +\frac{\sqrt{3}}{2}}$$

$$x_1 = y_1 = z_1 = \frac{1}{\sqrt{3}}$$

$$\boxed{\lambda = -\frac{\sqrt{3}}{2}}$$

$$x_2 = y_2 = z_2 = -\frac{1}{\sqrt{3}}$$

Vidimo da je maksimalna vrijednost za f jednako

$$f(T_1) = \frac{3}{\sqrt{3}} \quad ; \quad T_1\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

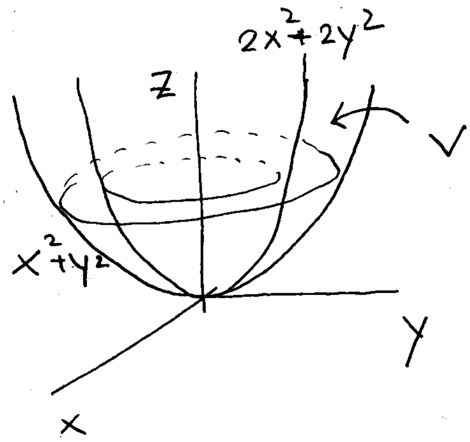
točka maksimuma

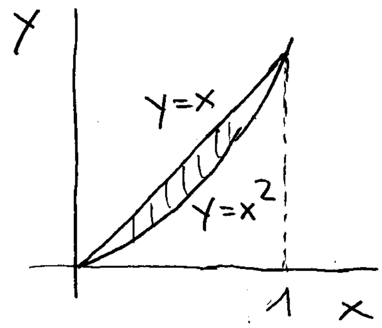
a minimalna

$$f(T_2) = -\frac{3}{\sqrt{3}} \quad ; \quad T_2\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$$

točka minimuma

$\boxed{Z6}$





Volumen je

$$V=\int_0^1 dx\int_{x^2}^{x} dy\int_{x^2+y^2}^{2x^2+2y^2} dz=\int_0^1 dx\int_{x^2}^{x} dy\,(\underbrace{2x^2+2y^2-x^2-y^2}_{x^2+y^2})$$

$$=\int_0^1 dx\left(x^2y+\frac{y^3}{3}\right)\Big|_{x^2}^{x}=\int_0^1 dx\left(x^3+\frac{x^3}{3}-x^4-\frac{x^6}{3}\right)$$

$$=\int_0^1 dx\left(\frac{4}{3}x^3-x^4-\frac{x^6}{3}\right)=\left(\frac{4}{3}\cdot\frac{x^4}{4}-\frac{x^5}{5}-\frac{x^7}{21}\right)\Big|_0^1$$

$$=\frac{1}{3}-\frac{1}{5}-\frac{1}{21}=\frac{35-21-5}{105}=\frac{9}{105}=\frac{3}{35}$$

$\boxed{Z8.1}$

$$J(a) = \int_0^{\pi/2} \ln(a^2 - \sin^2\theta)\, d\theta \; ; \quad a > 1$$

$$\frac{dJ}{da} = \int_0^{\pi/2} \frac{2a}{a^2 - \sin^2\theta}\, d\theta = [\text{ Bronštejn , str. 1035}]$$

$$= \frac{2a}{a\sqrt{a^2-1}} \arctan\left[\frac{\sqrt{b^2+1}\ \tan x}{a}\right]\Bigg|_0^{\pi/2}$$

$$= \frac{2}{\sqrt{a^2-1}} \left\{ \frac{\pi}{2} - 0 \right\} = \frac{\pi}{\sqrt{a^2-1}}$$

Integracijom se dobiva

$$J = \pi \ln(a + \sqrt{a^2-1}) + C$$

Kako odrediti konstantu C? Promotrimo granicu $a \to \infty$

$$J(a) = \int_0^{\pi/2} \ln(a^2 - \sin^2\theta)\, d\theta = \pi \ln(a + \sqrt{a^2-1}) + C$$

Napišimo

$$\pi \ln(a + \sqrt{a^2-1}) = 2\int_0^{\pi/2} \ln(a + \sqrt{a^2-1})\, d\theta$$

$$= \int_0^{\pi/2} \ln(a + \sqrt{a^2-1})^2\, d\theta$$

Odnosno,

$$C = \int_0^{\pi/2} \ln\left[\frac{a^2 - \sin^2\theta}{(a + \sqrt{a^2-1})^2}\right] d\theta$$

Za $a \to \infty$ imamo

$$C = \lim_{a \to \infty} \int_0^{\pi/2} \ln\left[\frac{1 - (\sin\theta / a)^2}{\left(1 + \sqrt{1 - 1/a^2}\right)^2} \right] d\theta$$

$$= -\ln\left(\frac{1}{2^2}\right) \cdot \frac{\pi}{2} = -\ln 2 \cdot \pi$$

Sve skupa

$$J(a) = \pi \ln\left(\frac{a + \sqrt{a^2 - 1}}{2}\right)$$