MATEMATIČKE METODE FIZIKE II

Drugi kolokvij 04.07.2014.

1. (a) Pokažite da su funkcije $f(z) = c$ i $f(z) = z$, gdje je c kompleksna konstanta, analitičke u cijeloj kompleksnoj ravnini.
(b) Zadane su analitičke funkcije $f_1(x,y) = u_1(x,y) + i\, v_1(x,y)$ te $f_2(x,y) = u_2(x,y) + i\, v_2(x,y)$ na nekom području u kompleksnoj ravnini. Pokažite da su tada i $f_1 + f_2$ i $f_1 \cdot f_2$ analitičke funkcije na istom području kompleksne ravnine.
(c) Pomoću (a) i (b) te matematičke indukcije, pokažite da je kompleksni polinom

$$f(z) = \sum_{k=0}^{n} a_k z^k$$

analitička funkcija u cijeloj kompleksnoj ravnini.

2. Riješite jednadžbu

$$\cos z = 2$$

3. Odredite imaginarni dio analitičke funkcije $f(z) = u(x,y) + iv(x,y)$ gdje je $u = \Phi(x^2 + y^2)$.
***Uputa*:** najprije nađite realni dio od $f(z)$, a nakon toga upotrijebite C-R uvjete.

4. Napišite Laurentov red oko $z = 0$ za navedenu funkciju i pri tom navedite o kakvom se singularitetu u točki $z = 0$ radi:

$$f(z) = e^{z^2} / z^3$$

5. Izračunajte realni integral pomoću teorema o reziduumima

$$\int_{-\infty}^{\infty} \frac{1}{(x^2 + a^2)(x^2 + b^2)}\, dx$$

gdje su a, $b > 0$.

$\boxed{1.}$

(a) Funkcija $f(z) = c$ zadovoljava C-R uvjete u svakoj točki kompleksne ravnine pa je onda i analitička u z-ravnini.

Funkcija

$$f(z) = x + iy$$
$$u = x$$
$$v = y$$

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \Rightarrow 1 = 1 \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \Rightarrow 0 = 0 \end{aligned}\right\} \text{zadovoljava!} \Rightarrow f = c \text{ i } f(z) = z \text{ su analitičke}$$

(b) Funkcije f_1, f_2 su analitičke

$$\frac{\partial u_1}{\partial x} = \frac{\partial v_1}{\partial y} \; ; \; \frac{\partial u_1}{\partial y} = -\frac{\partial v_1}{\partial x}$$

$$\frac{\partial u_2}{\partial x} = \frac{\partial v_2}{\partial y} \; ; \; \frac{\partial u_2}{\partial y} = -\frac{\partial v_2}{\partial x}$$

Funkcija $f_1 + f_2 = (u_1 + u_2) + i(v_1 + v_2)$

$$\frac{\partial}{\partial x}(u_1 + u_2) = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial x} = \frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial y} = \frac{\partial}{\partial y}(v_1 + v_2)$$

$$\frac{\partial}{\partial y}(u_1 + u_2) = \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial y} = -\frac{\partial v_1}{\partial x} - \frac{\partial v_2}{\partial x} = -\frac{\partial}{\partial x}(v_1 + v_2)$$

Funkcija $f_1 + f_2$ zadovoljava C-R uvjete pa je onda analitička u području u kojem su i f_1 i f_2 analitičke.

Funkcija $f_1 \cdot f_2 = (u_1 + i v_1)(u_2 + i v_2)$

$$= (u_1 u_2 - v_1 v_2) + i(u_1 v_2 + v_1 u_2)$$

$$\frac{\partial}{\partial x}(u_1 u_2 - v_1 v_2) = \frac{\partial u_1}{\partial x} u_2 + \frac{\partial u_2}{\partial x} u_1 - \frac{\partial v_1}{\partial x} v_2 - v_1 \frac{\partial v_2}{\partial x}$$

$$= \frac{\partial v_1}{\partial y} u_2 + \frac{\partial v_2}{\partial y} u_1 + \frac{\partial u_1}{\partial y} v_2 + v_1 \frac{\partial u_2}{\partial y}$$

$$= \frac{\partial}{\partial y}(u_1 v_2 + v_1 u_2)$$

$$\frac{\partial}{\partial y}(u_1 u_2 - v_1 v_2) = \frac{\partial u_1}{\partial y} u_2 + u_1 \frac{\partial u_2}{\partial y} - \frac{\partial v_1}{\partial y} v_2 - v_1 \frac{\partial v_2}{\partial y}$$

$$= -\frac{\partial v_1}{\partial x} u_2 - u_1 \frac{\partial v_2}{\partial x} + \frac{\partial u_1}{\partial x} v_2 - v_1 \frac{\partial u_2}{\partial x}$$

$$= -\frac{\partial}{\partial x}(u_1 v_2 + u_2 v_1)$$

C-R uvjeti su zadovoljeni pa $f_1 f_2$ analitička.

(c) Najprije treba pokazati da z^k analitička. Pretpostavimo da je z^{k-1} analitička. Funkcija z je analitička pa je prema (b) također i produkt $z \cdot z^{k-1} = z^k$ analitička.

Funkcija $a_k z^k$ je analitička jer je produkt konstante a_k i z^k.

Pretpostavimo da je $\sum_{k=0}^{n-1} a_k z^k$ analitička. Tada je

$$\sum_{k=0}^{n} a_k z^k = \sum_{k=0}^{n-1} a_k z^k + a_n z^n$$

analitička prema (b).

$\boxed{2.}$

$$\cos z = 2$$

$$\frac{1}{2}\left(e^{iz} + e^{-iz}\right) = 2$$

$$e^{iz} + e^{-iz} = 4$$

$$u = e^{iz}; \qquad u + u^{-1} = 4 \quad / u$$

$$u^2 - 4u + 1 = 0$$

$$u_{1,2} = +2 \pm \sqrt{4-1} = 2 \pm \sqrt{3}$$

$$iz = \mathrm{Ln}(2 \pm \sqrt{3})$$

$$\mathrm{Ln}(2+\sqrt{3}) = \ln(2+\sqrt{3}) + i\,2k\pi$$

$$\mathrm{Ln}(2-\sqrt{3}) = \ln(2-\sqrt{3}) + i\,2k\pi$$

$$z_1 = 2k\pi - i\ln(2+\sqrt{3})$$

$$z_2 = 2k\pi - i\ln(2-\sqrt{3})$$

$\boxed{3.}$

Označimo: $t = x^2 + y^2$

$$f = u + iv \;;\quad u = \Phi(x^2+y^2) = \Phi(t)$$

$$\frac{\partial u}{\partial x} = \frac{d\Phi}{dt}\cdot\frac{\partial t}{\partial x} = \Phi' \cdot 2x$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\Phi' \cdot 2x\right) = \Phi'' \cdot (2x)^2 + \Phi' \cdot 2$$

$$\frac{\partial^2 u}{\partial y^2} = \Phi'' \cdot (2y)^2 + 2\Phi'$$

$$\vec{\nabla}^2 u = 0 \quad \Rightarrow \quad 4\,\Phi'' \cdot (x^2+y^2) + 4\Phi' = 0$$

$$\frac{\Phi''}{\Phi'} = -\frac{1}{t}$$

$$\ln \Phi' = -\ln t + c_1$$

$$\Phi' = \frac{c_2}{t} \quad / \int$$

$$\Phi = c_2 \ln|t| + c_3$$

Pa je
$$u(x,y) = c_2 \ln|x^2+y^2| + c_3$$

Računamo funkciju v:

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -c_2 \frac{1}{x^2+y^2}\cdot 2y$$

$$v = -c_2 \cdot 2\cancel{y} \cdot \arctan\left(\frac{x}{y}\right)\cdot \frac{1}{\cancel{y}} + f(y)$$

$$\frac{\partial v}{\partial y} = -2c_2 \cdot \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \left(-\frac{x}{y^2}\right) + f'(y)$$

$$= 2c_2 x \frac{1}{x^2+y^2} + f'$$

$$= \frac{\partial u}{\partial x} = c_2 \frac{2x}{x^2+y^2}$$

$$\Rightarrow f' = 0$$

$$f = c_3$$

Konačno,

$$v(x,y) = -2c_2 \arctan\left(\frac{x}{y}\right) + c_3$$

Dodatak: nekim studentima je došao rezultat s

$$\arctan\left(\frac{y}{x}\right)$$

Zašto?

$$\arctan\left(\frac{x}{y}\right) + \arctan\left(\frac{y}{x}\right) = \underset{\substack{\uparrow \\ \text{ovo ovisi o} \\ \text{tome koliki su } x \text{ i } y}}{C} + \underbrace{\arctan \infty}_{\frac{\pi}{2}} = \text{konst.}$$

Dakle,

$$\arctan\left(\frac{x}{y}\right) = \text{konst.} - \arctan\left(\frac{y}{x}\right) \checkmark$$

$\boxed{4.}$ e^{z^2} oko $z=0$

$$e^{z^2} = 1 + z^2 + \frac{1}{2!} z^4 + \cdots \qquad |z| < \infty$$

$$f(z) = \frac{1}{z^3}\left(1 + z^2 + \frac{1}{2!} z^4 + \frac{1}{3!} z^6 + \ldots\right)$$

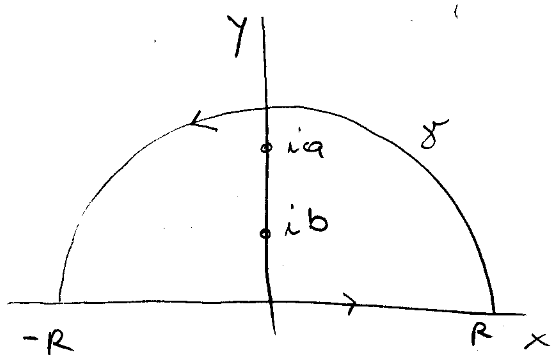
$$= \frac{1}{z^3} + \frac{1}{z} + \frac{1}{2!} z + \frac{1}{3!} z^3 + \ldots$$

Dakle, $z=0$ je pol 3. reda.

$\boxed{6.}$ Promatramo funkciju

$$f(z) = \frac{1}{(z^2+a^2)(z^2+b^2)}$$

U točkama $z = \pm ia$, $z = \pm ib$ su polovi 1. reda. Zanimaju nas polovi u gornjoj poluravnini.



$$\int_\gamma f(z)\,dz \xrightarrow[R\to\infty]{} 0$$

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = 2\pi i \sum_{z_K} \operatorname*{Res}_{z_K} f(z)$$

$$\operatorname*{Res}_{z=ia} f(z) = \frac{1}{2z(z^2+b^2)+2z(z^2+a^2)}\Big|_{z=ia} = \frac{1}{2ai(b^2-a^2)}$$

$$\operatorname*{Res}_{z=ib} f(z) = \frac{1}{2z(z^2+b^2)+2z(z^2+a^2)}\Big|_{z=ib} = \frac{1}{2ib(a^2-b^2)}$$

Imamo,

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = 2\pi i \,\frac{1}{2i(a^2-b^2)} \cdot \left(\frac{1}{b} - \frac{1}{a}\right)$$

$$= \frac{\pi}{a^2-b^2} \cdot \frac{a-b}{ab} = \frac{\pi}{ab(a+b)}$$