SVEUČILIŠTE U RIJECI, FAKULTET ZA FIZIKU

Diplomski studij Inženjerstvo i fizika materijala

Ak. god. 2021./2022.

TEORIJSKA FIZIKA I PRIMJENE II

Drugi kolokvij 9. 6. 2022.

ZADATAK 1 Čestica ima valnu funkciju

$$\psi(\mathbf{r}) = x^2 + y^2 - 2z^2 \ .$$

(a) Pokažite da je navedena funkcija, svojstvena funkcija operatora angularnog momenta $\mathbf{L}^2$ i L_z.
(b) Koliki su kvantni brojeve l i m za funkciju nađenu pod (a)?
(c) Nađite valnu funkciju koja ima isti kvantni broj l, a maksimalni broj m.

ZADATAK 2 Pretpostavimo da je čestica sa spinom 1/2 u stanju

$$\chi = \frac{1}{\sqrt{6}}\begin{pmatrix} 1+i \\ 2 \end{pmatrix} .$$

(a) Koja je vjerojatnost da mjerenjem S_z dobijemo vrijednost $+\hbar/2$, koja da dobijemo $-\hbar/2$?
(b) Koja je vjerojatnost da mjerenjem S_x dobijemo vrijednost $+\hbar/2$, a koja da dobijemo $-\hbar/2$?
(c) Izračunajte prosječnu vrijednost od S_x u stanju χ.

ZADATAK 3 Promotrite sljedeću varijacijsku probnu valnu funkciju za harmonijski oscilator frekvencije ω:

$$\psi = \begin{cases} C\left(a - |x|\right), & |x| \le a \\ 0, & \text{drugo} \end{cases} .$$

(a) Odredite vrijednost realnog parametra a kojim se energija osnovnog stanja minimizira.
(b) Kolika je procjena energije osnovnog stanja za danu probnu funkciju? Usporedite je s točnom energijom.
***Uputa*:** primijetite da je $\psi'' = C\delta(x + a) - 2C\delta(x) + C\delta(x - a)$.

ZADATAK 4 Elektron u vodikovom atomu nalazi se u osnovnom stanju. Odredite korekciju prvog reda za energiju osnovnog stanja ako na elektron djeluje mala smetnja oblika $H' = \alpha r^2$ gdje je α konstanta.

1.

$$\psi_{em} = x^2 + y^2 - 2z^2$$

$$= r^2 \sin^2\theta \cos^2\varphi + r^2 \sin^2\theta \sin^2\varphi - 2r^2\cos^2\theta = \underbrace{r^2 \sin^2\theta}_{1-\cos^2\theta} - 2r^2\cos^2\theta$$

$$= r^2(1 - 3\cos^2\theta)$$

$$= -r^2 \cdot \sqrt{\frac{16\pi}{5}}\, Y_{20}(\theta,\varphi)$$

$m = 0$

$\ell = 2$

Maksimalni $m = 2$ i odgovarajuće funkcije

$$c r^2 Y_{22} = c r^2 \sqrt{\frac{15}{32\pi}} \sin^2 e^{2i\varphi}$$

2.

$$\chi = \frac{1}{\sqrt{6}} \begin{pmatrix} 1+i \\ 2 \end{pmatrix}$$

(a) Razvijmo χ po χ_+ i χ_-; to su vlastiti vektori od S_z

$$\chi = c_+ \chi_+ + c_- \chi_-$$

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \; ; \; \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Treba odrediti $|c_+|^2$ i $|c_-|^2$. Pomnožimo sve s lijeva sa χ_+^+ ← transponirano konjugirano

$$\chi_+^+ \chi = c_+ \underbrace{\chi_+^+ \chi_+}_{(1 \;\; 0)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1} + c_- \underbrace{\chi_+^+ \chi_-}_{(1 \;\; 0)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0}$$

$$c_+ = (1 \;\; 0)\begin{pmatrix} 1+i \\ 2 \end{pmatrix} \cdot \frac{1}{\sqrt{6}} = (1+i) \cdot \frac{1}{\sqrt{6}}$$

$$P_+ = |c_+|^2 = \frac{2}{6} = \frac{1}{3}$$

Vjerojatnost da mjerimo $\hbar/2$ iznosi $1/3$.

$$c_- = (0 \;\; 1)\begin{pmatrix} 1+i \\ 2 \end{pmatrix} \cdot \frac{1}{\sqrt{6}} = \frac{2}{\sqrt{6}}$$

$$P_- = |c_-|^2 = \frac{4}{6} = \frac{2}{3}$$

Vjerojatnost da mjerimo $-\hbar/2$ je $2/3$.

(b) Razvijmo χ po vlastitim vektorima od S_x. Treba najprije naći vlastite vektore u bazi $\{\chi_+, \chi_-\}$ i vlastite vrijednosti od S_x.

$$S_x \doteq \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Problem vlastitih vrijednosti i vektora za S_x

$$\det(S_x - \lambda 1) = 0 \Rightarrow \det\begin{pmatrix} -\lambda & \hbar/2 \\ \hbar/2 & -\lambda \end{pmatrix} = 0$$

$$\Rightarrow \lambda^2 = \left(\frac{\hbar}{2}\right)^2, \; \lambda_{1,2} = \pm \frac{\hbar}{2}$$

Natrag u matricu $S_x - \lambda 1$

$$\lambda_1 = +\hbar/2 \qquad \begin{pmatrix} -\hbar/2 & \hbar/2 \\ \hbar/2 & -\hbar/2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \quad \Rightarrow \quad \begin{aligned} -\hbar/2\, v_1 + \hbar/2\, v_2 &= 0 \\ \hbar/2\, v_1 - \hbar/2\, v_2 &= 0 \end{aligned}$$

Dvije jednake jednadžbe. Rješenje:

$v_1 = 1;\ v_2 = 1$

$$\lambda_2 = -\hbar/2 \qquad \begin{pmatrix} \hbar/2 & \hbar/2 \\ \hbar/2 & \hbar/2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \quad \Rightarrow \quad v_1 \cdot \frac{\hbar}{2} + v_2 \frac{\hbar}{2} = 0$$

Rješenje:

$v_1 = 1,\ v_2 = -1$

U obliku jednostupčanih matrica (normiramo!)

$$|S_x;+\rangle \doteq \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$|S_x;-\rangle \doteq \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Razvijamo $|\chi\rangle$

$$|\chi\rangle = C_+ |S_x;+\rangle + C_- |S_x;-\rangle \quad / \langle S_x;+|$$

$$\langle S_x;+|\chi\rangle = C_+ \underbrace{\langle S_x;+|S_x;+\rangle}_{=1} + C_- \underbrace{\langle S_x;+|S_x;-\rangle}_{=0}$$

$$C_+ = \langle S_x;+|\chi\rangle = \frac{1}{\sqrt{2}} (1, 1) \begin{pmatrix} 1+i \\ 2 \end{pmatrix} \cdot \frac{1}{\sqrt{6}} =$$

$$= \frac{1}{\sqrt{12}} \left[(1+i) + 2 \right] = \frac{3+i}{\sqrt{12}}$$

$$P_+ = |C_+|^2 = \frac{10}{12} = \frac{5}{6}$$

Vjerojatnost da kad mjerimo S_x dobijemo $+\hbar/2$ je $\frac{5}{6}$.

$$C_- = \langle S_x;-|\chi\rangle = \frac{1}{\sqrt{2}} (1, -1) \begin{pmatrix} 1+i \\ 2 \end{pmatrix} \cdot \frac{1}{\sqrt{6}}$$

$$= \frac{1}{\sqrt{12}} \left[1+i-2 \right] = \frac{1}{\sqrt{12}} \left[i - 1 \right]$$

$$P_- = |C_-|^2 = \frac{2}{12} = \frac{1}{6}$$

Vjerojatnost da kad mjerimo S_x dobijemo $-\hbar/2$ je $\frac{1}{6}$.

(c)

$$\langle S_x\rangle = \chi^{+} S_x \chi = \frac{1}{\sqrt{6}}(1-i, 2)\,\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\frac{1}{\sqrt{6}}\begin{pmatrix}1+i\\ 2\end{pmatrix} =$$

$$= \frac{\hbar}{12}(1-i, 2)\begin{pmatrix}2\\ 1+i\end{pmatrix} = \frac{\hbar}{12}\left(2-\cancel{2i}+2+\cancel{2i}\right)$$

$$= \frac{4\hbar}{12} = \frac{\hbar}{3}$$

Mogli smo iskoristiti i rezultate dobijene pod (b) i

$$S_x|S_x;\pm\rangle = \pm\frac{\hbar}{2}|S_x;\pm\rangle$$

$$\langle S_x\rangle = \langle\chi|S_x|\chi\rangle = \left(c_+^*\langle S_x;+| + c_-^*\langle S_x;-|\right)S_x\left(c_+|S_x;+\rangle + c_-|S_x;-\rangle\right)$$

$$= \left(c_+^*\langle S_x;+| + c_-^*\langle S_x;-|\right)\left(c_+\cdot\frac{\hbar}{2}|S_x;+\rangle - c_-\frac{\hbar}{2}|S_x;-\rangle\right)$$

$$= \frac{\hbar}{2}|c_+|^2 - \frac{\hbar}{2}|c_-|^2 = \frac{\hbar}{2}\left(\frac{5}{6}-\frac{1}{6}\right)$$

$$= \frac{\hbar}{3}$$

3.

(a) Normalizacijska konstanta c

$$\int_{-a}^{a} |\psi|^2 dx = 1$$

$$c^2 \int_{-a}^{0} (a+x)^2 dx + c^2 \int_{0}^{a} (a-x)^2 dx = 1$$

$$c^2 \cdot \left[a^2 x + 2a \frac{x^2}{2} + \frac{x^3}{3} \right] \Big|_{-a}^{0} + c^2 \left[a^2 x - 2a \frac{x^2}{2} + \frac{x^3}{3} \right] \Big|_{0}^{a} = 1$$

$$c^2 \left[+ a^3 - a^3 + \frac{a^3}{3} \right] + c^2 \left[a^3 - a^3 + \frac{a^3}{3} \right] = 1$$

$$\frac{2c^2 a^3}{3} = 1 \implies c^2 = \frac{3}{2} \cdot \frac{1}{a^3}$$

Hamiltonijan za HO

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$$

Imamo,

$$\frac{p^2}{2m} \psi = -\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} = \frac{\hbar^2 c}{2m} \left[\delta(x+a) - 2\delta(x) + \delta(x-a) \right]$$

$$\langle H \rangle = \langle T \rangle + \langle V \rangle$$

$$\langle T \rangle = \int_{-a}^{a} \psi^* T \psi \, dx = \frac{\hbar^2 c^2}{2m} \int_{-a}^{a} dx (a - |x|) \left[\delta(x+a) - 2\delta(x) + \delta(x-a) \right]$$

$$= \frac{\hbar^2 c^2}{m} \cdot a = \frac{3\hbar^2}{2ma^2}$$

$$\langle V\rangle = \frac{m\omega^2c^2}{2}\int_{-a}^{0} x^2(a+x)^2\,dx + \frac{m\omega^2c^2}{2}\int_{0}^{a} x^2(a-x)^2\,dx$$

$$= \frac{m\omega^2c^2}{2}\left(a^2\cdot\frac{x^3}{3} + 2a\frac{x^4}{4} + \frac{x^5}{5}\right)\Bigg|_{-a}^{0}$$

$$+ \frac{m\omega^2c^2}{2}\left(a^2\cdot\frac{x^3}{3} - 2a\frac{x^4}{4} + \frac{x^5}{5}\right)\Bigg|_{0}^{a}$$

$$= \frac{m\omega^2c^2}{2}\cdot\left(+\frac{a^5}{3} - \frac{1}{2}a^5 + \frac{a^5}{5}\right)$$

$$+ \frac{m\omega^2c^2}{2}\left(\frac{a^5}{3} - \frac{a^5}{2} + \frac{a^5}{5}\right)$$

$$= m\omega^2c^2\cdot(10-15+6)\cdot\frac{a^5}{30}$$

$$= \frac{m\omega^2a^2}{20}$$

$$\langle H\rangle = \frac{3\hbar^2}{2ma^2} + \frac{m\omega^2a^2}{20}$$

$$\frac{d\langle H\rangle}{da} = 0 \Rightarrow \frac{3\hbar^2}{2m}\cdot(-2)\,a^{-3} + \frac{m\omega^2}{20}\cdot 2a = 0$$

$$\left(\frac{30\hbar^2}{m^2\omega^2}\right)^{1/4} = a_0$$

$$a_0^2 = \frac{\hbar}{m\omega}\sqrt{30}$$

Uvrstimo u $\langle H\rangle$

$$\langle H\rangle_0 = \frac{3\hbar^2}{2m}\cdot\frac{m\omega}{\hbar}\cdot\frac{1}{\sqrt{30}} + \frac{m\omega^2}{20}\cdot\frac{\hbar}{m\omega}\sqrt{30}$$

$$= \hbar\omega\underbrace{\left(\frac{3}{2\sqrt{30}} + \frac{\sqrt{30}}{20}\right)}_{0{,}55} = 0{,}55\,\hbar\omega > 0{,}5\,\hbar\omega$$

$\boxed{4.}$

Osnovno stanje: $\psi_{100} = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}$

Korekcija prvog reda:

$$\langle 100 | H' | 100 \rangle = \int \psi_{100}^{*} H' \psi_{100} \, dV$$

$$= \alpha \cdot \frac{1}{\pi a_0^3} \int r^2 e^{-2r/a_0} dV$$

$$= \frac{\alpha}{\pi a_0^3} \cdot \int d\Omega \int_0^\infty dr \, r^4 e^{-2r/a_0}$$

$$= \frac{4\alpha}{a_0^3} \cdot \frac{\Gamma(5)}{\left(\frac{2}{a_0}\right)^5} = \frac{\alpha a_0^2}{8} \cdot 4!$$

$$= 3 \alpha a_0^2$$